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**HERBERT  
GROSS:**

Hi, always being somebody who wants to get something for nothing, I figure any time in this course that we'll study the new function, the very next thing that we'll do is study the inverse function. In other words, since the inverse is just a change in emphasis, why not just change the emphasis to get the result that we want? So you see, today's lesson is called the inverse trigonometric functions. And the longest part of this lecture will be a few computations just to get the feel of things. The reason being that other than that, everything that we have to use will be drawn from results that we've already studied in our lessons called inverse functions. It's still the same idea, only a different illustration. Well, at any rate, let's get on with the subject.

Let's start with the curve 'y' equals 'sine x'. Now you see, when we plot the curve 'y' equals 'sine x', it doesn't take us very long to discover that this curve is not 1:1. In fact, it misses by a long shot. Because for every 'y' value between minus 1 and 1, there are infinitely many values of 'x' that produce that value of 'y'. This is an oscillating type of function. So you see from our rigorous point of view, there really isn't much sense in defining the inverse sine function.

Well, in a way, this is an artificial drawback that we have. Because you also recall that when we talked about functions that were not 1:1, we could always break them down into a union of 1:1 functions.

For example, notice in the part of the curve here which I've accented, namely, the domain from minus pi over 2 to pi over 2, notice that on that domain, the sine is 1:1. And it's certainly onto the interval from minus 1 to 1. In other words, every value of the sign is taken on once and only once on the interval from minus pi over 2 to pi over 2.

So you see, if I disregarded everything but this part of the curve, and how could I do that? What shall I name this function? Well, why not do something like this. Let me invent a new name. Let me call this curve "s sub 0" of x' where "s sub 0" of x' is defined to be 'sine x' provided that 'x' is in the closed interval from minus pi over 2 to pi over 2. Again, be very, very

careful. Recall that when we talked about defining functions, we said that to specify two functions as being equal it was not enough that the function machine was the same, the inputs had to be the same. In other words, two functions had to be equal. Or to be equal, the two functions had to have precisely the same domain. So notice in this sense there is a big difference between  $\sin^{-1} x$  and the function that's being called  $\sin x$ .

What is the big difference? Well, for the function that's called  $\sin x$ , the domain is all real numbers. So the function called  $\sin^{-1} x$ , the domain is only the closed interval from  $-\pi/2$  to  $\pi/2$ .

Now again, there's nothing sacred about picking this particular interval. What I could have done was define say, another curve, which I'll call  $\sin^1 x$ .  $\sin^1 x$  will be the function  $\sin x$ . But now, the domain will be shall we say, from  $3\pi/2$ , or from  $\pi/2$  to  $3\pi/2$ . In other words, notice that this portion of our sine curve is also 1:1, and runs the full gamut of values that the sine can take on from 1 to minus 1.

And by the way, I don't have to be prejudiced this way either. I could just have easily have worked with negative values. In other words, I could have invented say another curve, which I'll call  $\sin^{-1} x$ , where this subscript simply means I'm going in the opposite direction.

$\sin^{-1} x$  might just have well have been what? The curve  $\sin x$ . But now again, how are all of these things differing? Only in the choice of the domain. See now the domain would be from  $-\pi/2$  to  $3\pi/2$ .

Again, in terms of the picture, this portion here would be fine.

You see what I can do is break up this curve that fails to be single valued by-- not single valued. Fails to be 1:1 by a long shot, into a union of 1:1 curves. In fact, if I wanted to use some fancy mathematical language, which I'll write down just to look impressive here. But if it bothers you, ignore it completely. I guess what I'm really saying is, I could have defined  $\sin x$  to be the union  $\sin^n x$  as  $n$  goes from minus infinity to infinity.

Meaning what? Take all of these pieces with these integral subscripts and form their union, and that infinite union gives you back the entire curve. Well, what am I making all this issue about in the first place? You see, we can't have an inverse function unless our original function is 1:1. In fact, let's see what does happen.

Remember how we invert the function? In other words, how do you get from the graph of 'y' equals 'f of x' to the graph of 'y' equals 'f inverse of x'? Recalling from our previous lecture on this topic, what you essentially do, well, if you're good at visualizing, you just reflect with respect to the 45 degree line. If you're not so good at visualizing, what you do is, is what? You first rotate this thing through 90 degrees and then flop the thing over. And I've taken the liberty of doing that over here.

I take my curve and again, this would be very effective if I had overlays and wanted to slide these for you. I think it's something that you can see on your own. So all I'm doing is what? I rotate this thing through 90 degrees and then flop over the result. And now this curve will be the graph 'y' equals inverse sine of 'x' provided you can have an inverse. You see in the old math, this was fine because multi-valued functions didn't bother us at all. However, what is clear is notice that in this particular sense, the accented curve from before becomes this little region here. And notice that the function that I defined to be 's sub 0' does have an inverse. In other words, in terms of my new coordinate system, I can define what? The curve 'y' equals the inverse of 's sub 0'. The idea being what? That I cannot talk about the inverse sine.

Now the reason I bring this up is you'll notice that in this text, in most every text, people do use this notation. And they get around it in a rather cute way. As you read the text you will notice the phrase called principal values. Let me show you what I mean by that. What happens in the typical text is they'll say look it. By 'y' equals 'inverse sine x', we mean that 'x' equals 'sine y'. But that 'y' is restricted to be between the range of minus pi over 2 and pi over 2. This is called the range of principal values.

Now you see, this sometimes causes people to be a little bit upset because why do you have to make this restriction since nothing here seems to indicate that? All I hope that you can see from my discussion-- and this will also be written up in our supplementary notes, so if you need some reinforcement from what I'm doing in the-- not from what I'm doing the lecture, but in addition to what I'm doing in the lecture you can get this from the notes too. But what I want you to see is that what the average book defines to be 'y' equals inverse 'sine x', and then puts in the so-called principal values, simply turns out to be what I'm calling "'s sub 0' inverse'.

After all, what was 'y' equals "'s sub 0' inverse x'? That would be rewritten how? That's equivalent to saying 'x' equals "'s sub 0' of y'. But "'s sub 0' of y' was defined to be what? 'Sine y' provided that 'y' was in the range from minus pi over 2 to plus pi over 2. So far so good. At least that's one man's opinion.

What I'd like to show you now is again, the beauty of what inverse functions means. That from this point on, I can now, for example, in terms of calculus, get every single calculus result I need about derivatives of the inverse trigonometric functions just by restating them in terms of the ordinary trigonometric functions.

For example, suppose somebody were to say to me, find the derivative of inverse sine of 'x' with respect to 'x'. Again, keeping in mind now what this means, otherwise I don't have a function. I have to have this thing single-valued. This thing does it for me.

Look it, do I know how to find the derivative of the sine function? The answer is yes, I do. We've already done that. That was the last assignment, in fact. In other words, from 'x' equals 'sine y', can I find 'dx dy'? Certainly. 'dx dy' is 'cosine y'.

Now, look it. We didn't want 'dx dy', we wanted 'dy dx'. But since we're on a 1:1 strip here where the inverse function does exist, notice that the relationship between 'dx dy' and 'dy dx' is that they are reciprocals of one another. That was exactly one of the nice properties of this notation that we've talked about on several occasions before. In other words, 'dy dx' is '1 over cosine y'.

By the way, I like this better than saying things like the reciprocal of cosine or secant. I think when you print a book people like to use secant instead of 1 over the cosine simply because you can get everything on one line and don't have to write fractions this way. But that is not a major concern here. What I do want you to see is given that 'y' equals 'inverse sine of x', can I find dy dx? The answer is yes.

And by the way, notice that this is perfectly well defined. Namely, for a given value of 'y', 'cosine y' you see a single value. For a given number, it has only one cosine. You see, I don't need single-valuedness to get this result. The place I have to be careful about inverse functions existing is that since there are many different x-values that correspond to the same y-value, if I don't specify-- you see, if I have a multi-valued function, if I don't specify what branch I'm on the trouble will not come in when I'm looking for 'dy dx' at a given value of 'y'. It's that 99 times out of 100, when you're looking for 'dy dx', 'y' is given as a function of 'x'. In other words, you want to be able to convert this in an ambiguous way into a function of 'x'. That's the question that comes up. There is nothing wrong with this answer.

In other words, I can write down that 'dy dx' is '1 over cosine y'. But somebody can say to me,

I'd like the answer in terms of 'x'.

By the way, how can I get this answer in terms of 'x'? That's an interesting question sometimes. Recall that 'y' equals inverse sine of 'x'. The same as what? 'x' equals 'sine y'.

Now what identity do I know? I know that sine squared plus cosine squared is 1. 'x squared' is 'sine squared y'.

Now what do I know? I know that 'cosine squared y' is '1 - sine squared y'. So 'cosine y' is plus or minus the square root of '1 - sine squared y'. That's plus or minus the square root of '1 - x squared'.

Notice by the way, in this notation that 'y' wasn't any old number. 'y' had to be in what range? 'y' had to be between minus pi over 2 and pi over 2. And as long as that's the case, notice that in that range the cosine is positive. That makes the negative sign redundant. And by the way, that negative sign would not be redundant if we hadn't restricted our range to making the curve single-valued and 1:1. You see, it's the restriction that 'y' has to be between minus pi over 2 and pi over 2 that makes this the positive square root. In other words then, what we see is what? That therefore 'dy dx' is equal to '1 over cosine y'. That's 1 over the square root of '1 - x squared'. And that's a rather interesting result. It's a rather straightforward result. I would like to make a few comments about this thing.

And one of them is what I want to summarize with later too. Suppose you are given the problem of saying, I would like to trace a curve, or plot a curve, and all I know about that curve is that its derivative at any point is 1 over the square root of 1 minus the square of its x-coordinate. Do you have to know any trigonometry to understand that problem?

In other words, notice that this expression in no way utilizes trigonometry. Yet to solve this problem, it appears from what we've shown is that what? That given that the slope is this, the curve itself turns out to be 'y' equals 'sine inverse x'. And that is an inverse trigonometric function. And so here's a very important reason as to why the trigonometric functions are that important. The inverse trigonometric functions. Namely, inverse trigonometric functions can wind up as being what? The inverse derivative of a function which is non-trigonometric. And that's reason enough to study these things.

Again, as I told you in my last lecture, I was sadly mistaken when I thought that trigonometry belonged solely to the surveyor. I mean, look at all the different places that this material comes

up in. And if you don't like practical applications, at least observe that all we're really doing here is discussing inverse functions in terms of function itself. In other words, the result that we've shown now is that the integral of  $1$  over the square root of  $1 - x^2$  is 'inverse sine  $x$ ' plus a constant.

By the way, since there is such a connection between the inverse trig functions and the regular trig functions, and since the trig functions lend themselves to geometry rather nicely, I thought I would like to show you here a rather nice device that allows you say, to solve problems like this if you weren't given the answer in advance. In other words, you'll notice that in our problem by sort of working backwards, we found that the answer to this problem was 'inverse sine  $x + c$ '.

And the question is, what if we hadn't been given this? Is there a way that we could've utilized the knowledge of trigonometry, classical trigonometry, to get a hint as to how to do this? And not only is there such a way of doing this, but the method turns out to be so important that later in the course, in the section called 'techniques of integration', this comes up under the very special name of trigonometric substitution.

It works something like this. Whenever you see the sum or the difference of two squares, think of a right triangle. For example, in this case, if I call the hypotenuse  $1$  and one of the sides ' $x$ ', and call this angle  $\theta$  say, the third side of the triangle is the square root of  $1 - x^2$ .

And now if I think of this this way, what is the easiest relationship that allows me to express  $\theta$  as a trigonometric function involving ' $x$ '? There are many trigonometric relationships I can reach from this diagram. But it appears that the easiest one is the one that says 'sine  $\theta$ ' equals ' $x$ ' over  $1$ . In other words, 'sine  $\theta$ ' equals ' $x$ '. And now you see from this using differential notation and the like-- and again, the technique will be drilled into you and you'll get plenty of opportunity for using this in the exercises in the text. But for now, I just want you to get to see an idea of how the trigonometry does come back into this.

Notice that from this diagram I get this relation. Now taking the differential of both sides and notice that I'm working with the ordinary trig functions here, which I allegedly know at this stage. I also know from this triangle that the square root of  $1 - x^2$  is 'cosine  $\theta$ '. Therefore, if I make these substitutions in the integral ' $dx$ ' over the square root of  $1 - x^2$ ', using the differential notation ' $dx$ ' becomes replaced by 'cosine  $\theta$   $d\theta$ '. The square root of  $1 - x^2$  becomes replaced by 'cosine  $\theta$ '. These cancel.

The integral then turns out to be what See the integral of 'd theta' is 'theta' plus a constant. But what was 'theta'? Theta was what? 'Sine theta' was 'x', so 'theta' is the number whose sine is 'x'. Notice how the trigonometry comes in and helps us to solve a particular problem. A particular problem that might not have seemed that obvious if we hadn't have used the trigonometry.

And again, let me point out where we've used the fact that we're using 1:1. You see, when we drew this particular triangle, we assumed that 'theta' was in the first quadrant. The angle could have been any place if we're thinking of it as an angle. And knowing the sine only determines the cosine up to a plus or minus. In other words, technically speaking, this should've been plus or minus over here. But the fact that for principal values 'theta' must be between minus pi over 2 and pi over 2, and since the cosine is positive in that range that was why we were able to get rid of the negative sign.

So again, all I want you to see from this is the fact-- and this is very important. That aside from the inverse trigonometric functions being rather important, we can study them completely by giving a short lecture because everything that we have to know primarily came from previous lectures. The hard part as is often the case with inverse function notation is that you may not be familiar with the language that rapidly.

I sometimes, when I was first learning, had to think twice about notation like this. Let me start something new here. I'll start at the bottom and work up. When people wrote down identities like this, I found it very difficult to think in terms of those. What does this say? It says that the 'inverse cosine of x' is pi over 2 minus the 'inverse sine of x'. And I couldn't remember why that would be true. Yet the funny part is, if I let-- see let's go back to the top now and start. If I let 'y' equal 'inverse cosine x' and tried to draw what that really says-- see again, I have to be careful about 1:1 and what have you. But if I just mechanically translated this, this would say what? 'x' equals 'cosine y'. 'Cosine y' is 'x'. That's 'x' over 1. I've written it that way.

Well, look it. If this angle is 'y', it's kind of clear that this angle is 'pi over 2' minus 'y'. Now in this familiar environment, how difficult is it for me to see that the sum of these two angles is pi over 2? You see, in that familiar environment it's almost so obvious I wonder why anybody would want to point it out to me. Yet notice that from here, how do I read this? See what's another name for 'y'? In the angular system here, 'y' is that angle whose cosine is 'x'. It's 'inverse cosine x'.

What's 'pi over 2' minus 'y'? 'pi over 2' minus 'y' is that angle whose sine is 'x'. See notice that the side that's adjacent to 'y' is opposite 'pi over 2' minus 'y'. At any rate, 'pi over 2' minus 'y' can be named 'inverse sine x'. And so from this simple thing, simple because the language is familiar to me, I get down to the result that inverse 'cosine x' plus 'inverse sine x' is pi over 2. From which, of course, the step here is a triviality. See again, when we often deal with inverses, one of the fringe benefits that we have is that when we get stuck, we can always reduce the given result. Reverse the terminology so to speak and return from the inverse language to the original language.

Now again, let me point out that once you have a result like this all the results of calculus work the same way as before.

For example, suppose somebody says to me, gee, I wonder what the derivative of 'inverse cosine x' is with respect to 'x'? To be sure, I could go all through this again and mimic the results of getting the derivative for the inverse sine. But notice now, by this result, this is just the derivative with respect to 'x' of pi over 2 minus the 'inverse sine of x'. But this is the derivative of a difference. And a derivative of a difference is the difference of derivatives. And pi over 2 is a constant. So the derivative of pi over 2 is 0. I already know how to differentiate 'inverse sine of x', I've done that before. So I want what? Minus that derivative. I want 'minus 'd dx' 'inverse sine of x''. And that in turn is just what? Well, I'll write it over here. It's just minus 1 over the square root of '1 - 'x squared''. And again, another result but obtained with a minimum of new knowledge.

Now, I could go on but I think that from here on in, it's much easier for you to dig out what you want on your own. I had some other material that I thought I would give you, but I think that this will just turn out to be a little bit on the boring side now. Not boring, but in the sense that either you see what I'm driving at or you don't. And what I'd like you to do now is simply to go and see how much of this material on inverse trigonometric functions is yours now free of charge.

If I have to pick one thing I want to caution you about, don't be upset by the language called 'principal values' and what have you. Everything comes out in the wash if you recognize that we legally cannot define an inverse function unless we have a 1:1 function to begin with. So we must take the multi-valued trigonometric function and view it as a union of 1:1 curves. And then you see if you want to pick a different principal value, I think if you understand what's happening basically, you'll be able to do this thing on your own. In any event though, I think

this is enough on what we call the 'inverse circular functions'. So until next time, goodbye.

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