HERBERT GROSS: Hi, our lecture today is entitled inverse functions, and it's almost what you could call a natural follow-up to our lecture of last time when we talked briefly about 1:1 and onto functions. Inverse functions have a tremendous application as we progress through calculus, but of even more exciting impact is the fact that inverse functions are valuable in their own right. They are a pre-calculus topic. In fact, they appear as early in the curriculum as approximately the first grade.

See roughly speaking, inverse functions in plain English, mean that all we've done is made a switch in emphasis. Let's take a look at that.

Let's go back roughly to our first grade curriculum when one learns that 2 plus 3 equals 5, or that 5 minus 3 equals 2. Both of these statements say the same thing, but with a change in emphasis. It's as if 2 is being emphasized here while 5 is being emphasized here. This is rather interesting you see. For example, in the new mathematics, one talks about subtraction being the inverse of addition. And this is the same inverse that we want to talk about today as it applies to mathematics in general.

What do we that subtraction is the inverse of addition? It may sound fancy, but all it means is that if you know how to add, if you define subtraction properly, you automatically know how to subtract. And this, of course, is what's prevalent in the old change-making technique of going into a store, making a purchase, paying for the purchase, and when you receive your change, the clerk rarely, if ever, performs subtraction. You may recall that what he does is he adds onto the amount of the purchase the amount necessary to make up the denomination of the bill with which you paid him.

In other words, what we're saying here is, for example, that one may think of 5 minus 3 as being what? That number, which must be added onto 3 to give 5. You see, in this sense, subtraction is the inverse of addition. Once we know how to add, we automatically know how to subtract.

Now you see, this idea goes with us. We learned that multiplication and division are inverses of one another. And as we go on through higher mathematics, even on the pre-calculus level, we find additional examples of this.

For example, when one knows how to use exponents, one automatically knows how to study logarithms. Namely, if y equals the log of x to the base b, this is a synonym for saying that b to the y equals x.

And what is the basic difference between these two statements? They are paraphrases of one another. In one case it seems that the number y is being emphasized and in the other case it's the number x that's being emphasized.

And as in the case of most examples of paraphrasing, which of the two forms we use depends on what problem it is that we're trying to solve. In other words, if we know one of these two, then we automatically can study the other in terms of the one with which we feel more familiar. This, of course, continues when one gets to trigonometry and studies the so-called inverse trigonometric functions. If y equals the inverse sine of x that's the same thing as saying that x equals sine y.

Again, what is the basic difference? In one case, it seems that y is being emphasized. In the other case, it's x that's being emphasized. In terms of the usual calculus jargon of independent variable versus dependent variable it appears that what? In one case, x is the independent variable, y the dependent variable. In the other case, y is the independent variable, x the dependent variable.

To generalize this result what we're saying is simply this. If y equals f of x, you see where y is being emphasized, the dependent variable. If we wish to switch the emphasis, then we write x equals f inverse of y. This is read f inverse of y and it's the function which in a sense, inverts the roles of x and y. Let's see what this means more explicitly in terms of a particular example.

Let's suppose that we're given the equation y equals 2x minus 7. Or to represent this somewhat more abstractly, y equals f of x where f of x is 2x minus 7. Now you see, without mentioning the word inverse function, it turns out that early in our high school career we were finding inverse functions as soon as, for example, someone were to give us this problem and say solve for x in terms of y. Solve for x in terms of y. You see, if we solve for x in terms of y, we now have what? x is being emphasized. These two statements tell us the same thing. But now we write what? x equals f inverse of y. And f inverse of y is just y plus 7 over 2. Notice again the connection between f inverse and f. How one undoes the other.

In terms of our function machine idea, what we're saying is we may visualize the f machine whereby the input is x and the output will be twice x minus 7. In other words, the output will always be twice the input minus 7.

Now the question that comes up is, suppose we reverse the roles of the output and the input. In other words, suppose now we let the input be y, what will the output be? If we reverse the terminal so to speak, what we have shown is that now the f inverse machine would be what? The input is y, the output is y plus 7 over 2. And by the way, a question that we shall come back to very shortly that plays a rather important role here and which I'll emphasize from another point of view is that if you get into the idea of always wanting to call the input x and the output y, which is how we get geared to do things in terms of calculus. x is always the horizontal axis, y the vertical axis, and we always agree to plot the independent variable along the x-axis. In other words, the input along the x-axis, the output along the y-axis. Then the question is, could we have called this x and called this x plus 7 over 2? And we'll go with this in more detail in a little while. But obviously, what we call the name of the input should not affect how the machine behaves.

By the way, as a little aside, I thought it might be interesting to show why we use such notation as f inverse. f to the minus 1. It's rather interesting here.

Let's suppose we let a number go into the f machine. Call that number c. Notice that any number that goes into the f machine has as its output twice that number minus 7.

Suppose we now let that number be the input of the f inverse machine, what does the f inverse machine do? It adds 7 onto any input and then divides that result by 2. In other words, if we now run 2c minus 7 through the f inverse machine, we have 2c minus 7 plus 7 over 2 equals c. In other words, notice how the f inverse machine undoes the f machine.

If we wanted to use the language of last time in terms of composition of functions, what we do is what? What we're saying is that if you combine f followed by f inverse, f inverse following f, that that gives you what we can call the identity function. The identity function. Namely, if the input is c, the output will again be c. In other words, f inverse of f of c is just c back again.

In a similar way, notice that we can reverse these roles. We saw last time that composition of functions depends on which order you combine the functions. But notice that if you run d through the f inverse machine, the output will be d plus 7 over 2.

If this becomes the input of the f machine, remember what the f machine does. It doubles the input and subtracts 7. In other words, again, f of f inverse of d gives me d back again. In other words, in terms of composition of functions, f followed by f inverse or f inverse followed by f is what we call the identity function. That one is truly the inverse of the other from that particular point of view.

However, let's correlate what we're talking about now with the circle diagrams that we used in our last lecture. You see, first of all, let's recall that unless our function is both 1:1 and onto, we do not have an inverse function. Namely, for example, if our function had not been onto, then when we-- see, here's the idea again. Let me make sure this is clear. To get an inverse function, essentially all we do is this. If f is a function from A to B, the inverse function is defined by reversing the input and the output. Which means in terms of this diagram, we reverse the sense of our arrows. We reverse which end the arrowhead goes on. And what we're saying is, if we had a function from A to B, which was not onto, then you see when we reverse the arrowheads, f is not defined on all of b. In other words, the domain of f, the domain of the inverse function, would not exist because it would not be defined on all of B.

Secondly, if two different elements of A went into the same element of B when we reversed the arrowheads, the resulting function would not be single-valued. And hence, in terms of modern mathematics, it would not be a well defined function. So in other words, for the inverse to exist it must be that the original function is both 1:1 and onto. And as an example of that, this is what this diagram here represents.

And to make sure that we can read this all right, I have singled out a typical element of capital A, a typical element of capital B. Remember what our notation is. The notation is that f of a equals b. That the image of a under f is b. And to use that in terms of the inverse language, if I called g the function that I get when I reversed the arrowheads, g of b equals a. And g is what I'm calling f inverse. By way of further review, the domain of f is equal to the image of f inverse. And that's A. The image of f is the same as the domain of f inverse. And that's B.

And now, what the question is, is this. Notice that as long as you want to use the same diagram, all we have to do to express f inverse in terms of f is sort of to reverse the arrowheads. The question that comes up is, suppose you insist that the domain be listed first.

In other words, when we're going to talk about g or f inverse in this case, that's a function from B to A. So why don't we list B first

And you see again, we can do this. Here are the elements of B, here are the elements of A. And all we have to do is see what happens over here. For example, if we come back to here notice that the first element listed in B comes from the third element listed in A. So when I make up the inverse function, I just capitalize on this by writing the same thing. The only problem is-- and this is going to become a crucial one-- is the fact that if somehow or other you couldn't see these labels, if you couldn't see these labels and all you knew was that the first set was called the domain and the second set was called the image. If you now looked at these two functions, you see they wouldn't look anything at all alike. In other words, f and f inverse, while not independent of one another, do look quite different. For example, notice that f inverse causes the first two elements in here to sort of crisscross as they have images, and the second two elements of here crisscross. Notice though in terms of f, it's the second and third that crisscross and the first and fourth that don't intersect at all this way.

In other words, if you look at this curve or this diagram and compare it with this diagram, notice that there is a difference in what seems to be going on. Well again, this is quite abstract. Let's try to relate this as much as possible to the language of calculus and our coordinate geometry graphing techniques.

To begin with, let's suppose that we have a function f whose domain is the closed interval from a to b and whose range is the closed interval from to c to d. And the question that we'd like to raise is, under what conditions will f possess an inverse function? What does onto mean? What does 1:1 mean and things of this type?

Well, the first thing I'd like to point out is that if the graph y equals f of x looks something like this. See, notice that the domain is from a to b. The image is from c to d. Notice the fact that we have a break in the curve over here, tells us that our function is not onto. Namely, given this number p, which is in our image between c and d, there is no element of the domain that maps into p. So, in other words, if there is a break in the curve, the function is not onto and hence, it will not have an inverse.

Now suppose there is no break in the curve. Let's suppose now that the curve doubles back. It comes up and doubles back. Now what my claim is is that the function will not be 1:1. Well, how can we see that? Pick any part where the curve doubles back, pick a point like this in that range. Call that point q. Noticed that q is in the proper range of f now. y equals f of x.

Now the question is, given the y value of q, are there any x-values that map into q under y? And the answer is yes, there are. In fact, there are more than one. Namely, notice that both f of x1 and f of x2 equal q. In other words, in this case, f of x1 equals f of x2, even though x1 is unequal to x2. That means that this function is not 1:1. And because it's not 1:1, it doesn't have a well defined inverse function. Well, putting these two cases together, what it means for the function to be onto, what it means for the function to be 1:1, it turns out that if our curve is unbroken, then the only way our function can have an inverse function is that the curve must either always be rising or always be falling, and it can't have a break in it.

And by the way, as an aside, let me point out here the difference between a continuous variable-- meaning one that's defined on a whole interval-- and a discrete variable-- meaning where you get isolated pieces of data.

Notice, for example, if you plot y versus x the way we do in a lab experiment where for a particular value of x, you measure a value of y. Notice that the data can double back without the function being multi-valued. In other words, notice for example, that even though the curve doubles back here-- I can't call it a curve. The data doubles back.

Notice, for example, that no two different pieces of data have the same y-coordinate. In other words, given this point here as being q, there is only one piece of data that has its y-coordinate equal to q. However, of course, keep in mind it is possible that another piece of data will have the same coordinate. All I'm saying is that the idea of whether the curve always has to be rising or following certainly depends on whether you have a continuous curve or not.

Well, again, let's continue on with what inverse functions are all about. You see, this comes up with our whole idea of why do we make fun or why do we minimize single-valued functions in calculus? And the answer is that single-valued-- I'm sorry. Why do we always stick to single-valued functions and do away with multi-valued functions? And the answer is if you have a smooth curve, we can always break down a multi-valued function into a union of single-valued functions.

For example, if we take the curve c here as being y equals f of x, which plots like this. Notice if I take the points at which I have vertical tangents and break the curve up at those particular points. In this case, I'll get what curves? c1, c2, and c3. Notice that c is the union of c1, c2, and c3, but that each of the curves c1, c2, and c3 are either always rising or always falling.

In a similar way, when we have a function which doubles back-- and by the way, notice what the connection is between multi-valued and not 1:1. You see, notice that in terms of a function versus its inverse function idea, that if a function is multi-valued the inverse function cannot be 1:1. In other words, the idea being that when you interchange the domain and the range, sort of the curve flips over idea, all I want you to see here is that what? If you're given a function which is not single-valued, if we take the points at which horizontal tangents occur and break down the curve like this, we can break the curve down into a union of 1:1 functions. The hardship being of course, that when you start with a point like this, analytically speaking, it's rather difficult unless you invent some scheme to know which of the points here you want to single out.

In terms of our previous experience, it's sort of like saying to a person, I am thinking of the angle whose sine is 1/2. There are, you see, infinitely many functions whose sine is equal to 1/2.

Of course if we say to the person, I am thinking of the angle whose sine is 1/2 and the angle is between minus 90 degrees and plus 90 degrees, then the only possible answer is the angle must be 30 degrees. But notice that when you have a function which is not single-valued, the inverse will be a multi-valued function. And we'll talk more about that in a little while. Again, I just want to keep this shotgun approach going on just what an inverse function is in relationship to the function itself.

Again, let's look at this more abstractly. Here I have drawn a curve which is continuous and always rising. So I can talk about the inverse function. If the equation is y equals f of x, the inverse is written what? x equals f inverse y. And if this seems a little bit too abstract for you, think of a concrete representation.

Suppose the curve happened to represent y equals 10 to the x. Then another way of saying the same thing would be x equals log y. The convention of course here being that you don't usually write base 10. But we won't worry about that. You see, this is what? Two different ways of expressing the same curve. Whether I write y equals f of x or x equals f inverse y, I have the same curve this way. In terms of our arrows, you see what I'm saying is, if I start with x1, by going this way, my function determines the output y1. Inversely, if I start with y1 and reverse the arrows, I wind up with x1. Again, the basic difference being as to which of the two variables is being emphasized.

What the real problem is, is that most people say look it. I'm not used to studying curves this way. I'm not used to looking at the input being along the vertical axis and the output along the horizontal axis according to the way I've been trained when we're studying the inverse function. In other words, when y is the input, aren't we used to having y over here and then plotting the output along this axis? In other words, the question is given this graph, how do you arrive at this one?

You see, somehow or other, let's observe that if all you did was switch your orientation and say let me switch this by 90 degrees, notice that we would be in a little bit of trouble. In other words, if we start with this kind of a set up and we say, let's rotate through a positive 90 degrees. Notice now what we would wind up with is what? Our x-axis would be the way we want it, but the y-axis would now have the opposite sense of what we usually want our input axis to look like.

So after we rotate through 90 degrees, it would seem that the next step is to do what? Flip with respect to the x-axis. That means fold this thing over. In other words, a 90 degree rotation followed by a folding over gives me the orientation if I insist that the input has to be along the horizontal axis and the output along the vertical axis. What I want you to also notice though, is that if we don't insist on this, there is no reason why we have to use two separate diagrams. Again notice, these are two different ways of giving the same -- two different equations for giving the same curve. It's only when we want to switch the role and make sure that the input is along the horizontal axis that we have to go through this kind of a process. Let's look at this a little bit more concretely. What I call a semi-concrete illustration.

What I'm saying now is let's suppose this is the curve I've drawn in here, y equals f of x. Another way of saying that is x equals f inverse of y. And the question is, suppose I now want to plot this same curve, same equation, but now with the y-axis as my horizontal axis. You see again, in terms of geometry, how I shift my axes will not change this equation. But what the picture of this equation looks like will certainly depend on how I orient my axes.

So the idea is what? I simply fold this, rotate this thing, through 90 degrees. And if I do that, the resulting picture looks like this. And once the picture looks like this, the next step is what? Flip this with respect to the x-axis. And now my picture looks like this. In other words, x equals f inverse of y here and x equals f inverse of y here are the same equation. The reason that the picture looks different is because I didn't allow myself to use this as the axis of inputs here. In other words, again, as soon as I wanted to make this axis orient so it would be the horizontal axis, this is what I had to go through over here.

Now you see, the next refinement is that a person says look it, I'm not used to calling this the y-axis. What's in a name? Why don't we always agree to call the horizontal axis the x-axis and the vertical axis the y-axis? And if I agree to do that, notice what happens just by changing the names of the variables. All that happens is, is that now this becomes y equals f inverse of x. This is an important thing to notice then.

In other words, if you insist that the horizontal axis in both cases will be called the x-axis and the vertical axis the y-axis, then this would be the curve y equals f of x and this would be the curve y equals f inverse of x. But again, the whole thing comes about only when you insist on how you want your axes oriented. Let's go back to our problem of y equals 2x minus 7 and see what this thing means in terms of a graph.

As we saw previously, if y equals 2x minus 7, x is equal to y plus 7 over 2. And the idea is what? Let's see what this thing really means. If I plot the straight line y equals 2x minus 7, this is the line that I get. Notice that as long as I'm going to use the same orientation of axes here, it makes no difference whether I call this line y equals 2x minus 7 or whether I call it x equals y plus 7 over 2. They are two different names for the same line. The problem occurs when I insist that the independent variable always be plotted along the x-axis, the horizontal axis, and the dependent variable along the vertical axis.

Again, going through what we did before, I first take this thing and I rotate it through a positive 90 degrees. That takes this picture and transforms it into this one. I now take this and I flip it with respect to the x-axis, and that gives me this picture here.

Now what is this line here? It's x equals y plus 7 over 2. Again, this is the same equation as this one. The reason that the pictures look differently is the fact that we have changed the orientation of the axis. Again, if we now say OK, let's rename this the x-axis, let's rename this the y-axis, then this becomes what? y equals x plus 7 over 2. It's in this sense that we call this curve of this equation and this equation here, that these two equations are inverses of one another.

Again, in terms of what we said before, if you pick a particular value of x and compute y this way, then you apply this recipe to that. In other words, if you now take twice y, twice the output here, and subtract 7, your original input returns. In other words, this works exactly the same as we did before. But again, the whole basic difference is what? How you want to orient your axis. That the curves look different because your coordinate system is different.

Of course, the interesting question now is, if we compared these two curves, since the-- see, granted that the function and its inverse are different functions, they are somewhat related. They're not random. How are these two graphs related? That might be the next natural question to ask.

If we do that, the idea is simply this. Let's suppose we have y equals f of x as one of our curves. The curve happens to be invertible, meaning that f is always rising and it's unbroken, et cetera. The question is, if we now try to plot y equals f inverse of x in the same diagram. See notice now what I'm saying. In other words, I am not saying x equals f inverse of y here. I'm saying suppose I have the curve y equals f of x and also the curve y equals f inverse of x. How do these two curves look with respect to an x- and y-coordinate system?

See, let me do that part more slowly again. Let me come over here for a moment. Notice that y equals 2x minus 7 was my original curve in what I dealt with before. If I want to keep the same orientation of axis, the inverse function we saw was y equals x plus 7 over 2. The question that we're asking quite in general, not in this specific case, is how are these two curves related? And the solution goes something like this. Let's suppose that the point x1, y1 belongs to the curve c1. By definition of c1, that says that y1 is f of x1. By definition of inverse function, see if y1 is f of x1, that means if you interchange the input and the output, that's another way of saying what? That x1 is f inverse of y1. In other words again, if f maps x1 into y1, f inverse maps y1 into x1, by definition.

Now you see, compare this with the curve c2. See, this says what? That the input y1 maps into the output x1. In other words, notice that if you look at the f inverse situation here, when the input is y1, the output is x1. That's another way of saying that y1 comma x1 belongs to the curve c2. In other words, if x1, y1 belongs to y1 equals f of x1, then y1, x1 belongs to y equals f inverse of x.

Now, what is the relationship between the point x1 comma y1 and the point y1 comma x1? If we draw this little diagram, we observe that we have a couple of congruent triangles here. This length equals this length. This angle equals this angle. And this gives me a hint. This makes triangle OPQ isosceles. I draw the angle bisector of angle O. The angle bisector of an isosceles triangle is the perpendicular bisector of the base. And angle bisector of the vertex angle is the perpendicular bisector of the base. Well, you see that makes this angle equal to this angle. That makes this a 45 degree angle. In other words, the line that I've drawn is indeed, y equals x. And notice that P and Q are symmetrically located with respect to the line y equals x. In other words, going back to our original problem here, the curve c1 and c2 are related by the fact that they are mirror images of one another with respect to the line y equals x. That's exactly what I've drawn over here.

In other words, going back to the problem of how are the curves y equal 2x minus 7 and y equal x plus 7 over 2 related, the answer is they are mirror images of one another with respect to the line y equals x. They are symmetric with respect to that line.

Now you see, let's talk about this from another point of view also, and show what the tough thing is. You see, so far my whole discussion seems to have hinged on the fact that we have a function, which is invertible. What if you have a function which is non-invertible? Going back to something more familiar, why do we, talk about -- when y equals the square root of x, why do we have this convention that we take the positive square root? After all, doesn't the square root of x and minus the square root of x have the property that when you square them you get the same result? Plus or minus squared is always plus. And the answer is that if you square both sides here and think of this as being the curve y squared equals x, what happens is you get a multi-valued function. One value of x yields two values of y.

And the way we get around that is we break this curve down into two pieces, c1 and c2, where c1 is always rising. c2 is always falling here. In other words, we broke this thing off at the point of vertical tangency. And we can now think of this curve as being the union of two curves. One of which is y equals the positive square root of x and the other is y equals the negative square root of x. Now the question is, what happens when you have a function which is not single-valued. In other words, let's just invert this one. Let's suppose we started with the curve y equals x squared.

You see, now for a given value of y, I'm in trouble. Because if y1 is positive, there are two different values of x which yield this particular result. In other words, both of these have the property that when you square them you get y1. And all we're saying is that in a problem such as this, we can study this curve as two separate pieces. Call one of these curves k1. That will be the curve y equals x squared, where x is non-negative. So this will be the curve k1. And call the other one k2, where k2 will be what? The same curve y equals x squared, but its domain is the negative values of x. In other words, k2 will be this one over here.

And now the point is, if we deal with either of these two pieces separately, we can talk about inverse functions. Now the point is, which of these two halves do we use? And this is where the word principal values comes in. And you see what I'd like you to keep in mind is this, a little cliche I've written down here. It's called misinterpretation versus non-comprehension.

If you don't understand what something means, there's no danger you're going to misinterpret it. The danger is when you think that you know what something means and you have the thing twisted around. You see, the idea is this. Let's go back to our old friend y equals sine x. Let's pick the value of y equal to 1/2 say. And now we say to the person the same problem as we asked before, find the angle whose sine is 1/2.

Well, the point is to find that angle. If I draw this particular line, I can find all sorts of candidates. The point is that what we tried to do instead is to say, OK, well now restrict the function. We'll break this down to be a union of several curves. In other words, it'll be this curve union this one. Union this one. Union this one, et cetera.

What do all of these separate pieces have in common? What they have in common is that what? They are onto the range from minus 1 to 1 and on that range they are also 1:1. 1:1 and onto. Every value of y between minus 1 and 1 is taken on along each of these pieces. And no value occurs more than once on any of these pieces.

The point is it's not so crucial whether you take this particular one or whether you take this particular piece. That's something that's sort of arbitrary. What we must do to avoid misinterpretation is unless otherwise specified we say look it, unless you hear from me to the contrary, let's always agree that this is the little piece of the curve that we're talking about, or this is the piece that we're talking about. But the idea being what? Unless you make such a restriction, we cannot talk about inverse functions. The idea being that for an inverse function to exist, we must be able to back map. We must be able to go from the value in the image to the value in the domain without any danger of misinterpretation.

We can conclude our example with returning to our y equals x squared idea again. You see the idea is given the curve y equals x squared, we can think of it in terms of our pieces k2 and k1 as before. The accented piece being k2, the non-accented piece being k1. And what we're saying is the inverse of k1 is this curve here, which I'll call k1 inverse. The inverse of k2 is this curve here, which I'll call k2 inverse. In other words, the important thing is I can find the inverse of either this curve or this curve. And in fact, how do I do that? Again, with respect to the 45 degree line, the line y equals x, notice that k1 and k1 inverse are symmetric with respect to this 45 degree line. And similarly, so are k2 and k2 inverse. The thing I must be very careful about and this is where problems occur, is I must not confuse-- for example, what I can't do is take, for example, k2 and k1 inverse. Notice the built-in idea here. These two curves together are not symmetric with respect to the 45 degree line.

You see what we're saying here is, is what? That for this particular curve, x and y are both positive. So obviously, anything that matches it must have x and y both positive. And that doesn't happen over here. What we're saying is you can't do these things completely at random. However, what you can do is either take k1 and match that with k1 inverse, k2 and match that with k2 inverse. It's not important which of the two ways you do this, as long as you understand that there is a danger of getting mixed up once the curve itself is not 1:1. In other words, when we break the curve down into 1:1 pieces, we have to make sure that we match these things up properly.

Now, this is all we're going to say about inverse functions for the time being. The rest will be taken care of in the exercises in this unit. However, we will return to this point very, very strongly later in our discussion of calculus. The important point to remember is that a function and its inverse function give us two different ways of expressing the same information. And that we can use whichever one happens to be to our advantage. Well, until next time, goodbye.

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