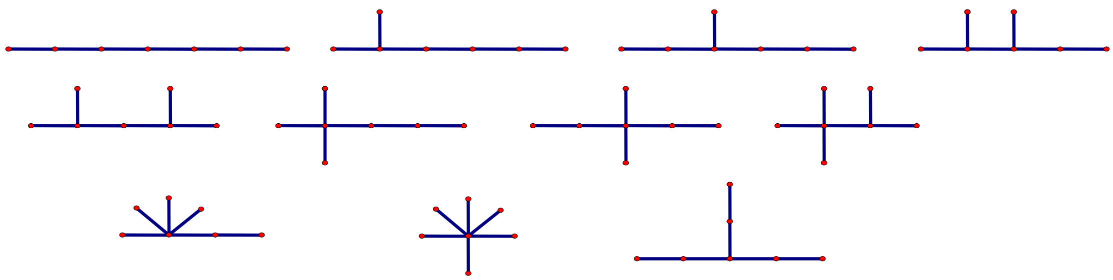
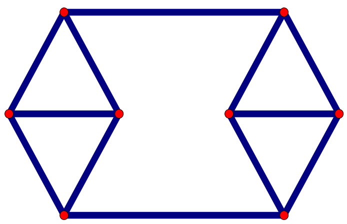
Combinatorics: The Fine Art of Counting

Week Eight Problems

1. Diagrams of all the distinct non-isomorphic trees on 6 or fewer vertices are listed in the lecture notes. Extend this list by drawing all the distinct non-isomorphic trees on 7 vertices.



2. Give an example of a 3-regular graph with 8 vertices which is not isomorphic to the graph of a cube (prove that it is not isomorphic by demonstrating that it possesses some feature that the cube does not or vice-versa).



Note that this graph contains several 3-cycles (triangles), whereas the cube does not, therefore the graphs cannot be isomorphic.

3. For each of the regular polyhedral graphs (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), try to answer the following questions:

- What is its degree and diameter?

Tetrahedron:	*degree 3, diameter 1*
Cube:	*degree 3, diameter 3*
Octahedron:	*degree 4, diameter 2*

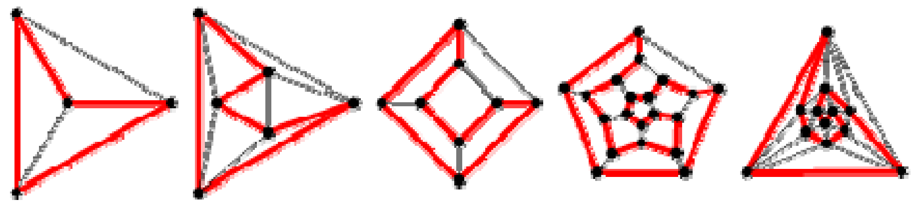
Dodecahedron: degree 3, diameter 5
Icosahedron: degree 5, diameter 3

- Is it Eulerian? If so, find an Eulerian circuit, otherwise prove it is not.

  Only the octahedron is Eulerian, since all the others have odd degree.

  Label the vertices of the octahedron to match a die, i.e. numbers from 1 to 6 with non-adjacent vertices summing to 7. One Eulerian circuit is: ***1-2-4-6-3-5-1-3-2-6-5-4-1****.*

- Can you find a Hamiltonian circuit?



4. Examples of the graphs of several saturated hydrocarbons are shown in the lecture notes - CH_4 (methane), C_2H_6 (ethane), C_3H_8 (propane), and others. The graph of a saturated hydrocarbon is always a tree where each hydrogen atom is represented by a vertex of degree 1 and each carbon atom is represented by a vertex of degree 4.

Given n carbon atoms, how many hydrogen atoms can be added to make up a saturated hydrocarbon, i.e. what chemical formulas for saturated hydrocarbons are possible?

$\boldsymbol{C_nH_{2n+2}}$

In the examples listed in the lecture notes, both butane and isobutene have the same chemical formula C_4H_{10}, but different molecular structures. Is there another saturated hydrocarbon besides propane with the molecular formula C_3H_8? How many different saturated hydrocarbons are their with 5 carbon atoms? Can you conjecture a general formula?

Propane is the only saturated hydrocarbon with the formula C_3H_8. There are ***3*** *different saturated hydrocarbons with the formula C_5H_{12}, corresponding to the three non-isomorphic trees with 5 vertices (note that all the vertices of these trees have degree less than or equal to 4). In general the number of different molecules with the formula C_nH_{2n+2} is equal to the number of non-isomorphic trees on n vertices with all vertices having degree less than or equal to 4 – these are called quartic trees. For n = 1,2,3,… the sequence of these numbers begins:*

1, 1, 1, 2, 3, 5, 9, 18, 35, 75, 159, 355, 802, 1858, 4347, 10359, 24894, 60523, …

There is no simple formula known for the numbers in this sequence. For more information about what is know, lookup this sequence on-line in Sloane's Encyclopedia of Integer Sequences:

http://www.research.att.com/~njas/sequences/

5. For each of the graphs K_5 and $K_{3,3}$ count the number of distinct Hamiltonian cycles they contain. For K_5 count the number of distinct Eulerian circuits.

*K_5 has 5!/(5*2) = **12** distinct Hamiltonian cycles, since every permutation of the 5 vertices determines a Hamiltonian cycle, but each cycle is counted 10 times due to symmetry (5 possible starting points * 2 directions).*

*$K_{3,3}$ has 6*3*2*2*1*1/(6*2) = **6** distinct Hamiltonian cycles –pick one of the six vertices to start at and then count the number of choices for each successive vertex and divide by 12 since each cycle will be counted 6*2=12 times due to symmetry.*

*If we do not distinguish the starting point or direction, the number of Eulerian circuits in K_5 is 12 + 120 = **132**. These can be counted by considering the decomposition of an Eulerian circuit on K_5 into cycles. Since there are 10 edges in such a circuit there are two ways this can be done, either as two 5-cycles, or as a two 3-cycles and a 4-cycle.*

*In the case of two 5-cycles, since we are not distinguishing starting point we will assume we begin the circuit by traversing one of the 5-cycles. There are 5! ways to choose the first 5-cycle, and 5 possible starting points, giving 5!/5 = 24 distinct possibilities. Once we have fixed these choices, there are exactly two possibilities for the second cycle – it must visit the vertices in an order which skips every other vertex relative to the first, and it can do so in one of two directions relative to the first (once we fix the orientation of the first cycle, we can distinguish orientations of the second). We could have chosen the two cycles in either order, so we obtain 24*2/2 = 24. Finally note that reversing the order of the circuit also results in a circuit which consists of a 5-cycle followed by a 5-cycle, so we divide by 2 again to get **12**.*

*In the case of two 3-cycles and a 4-cycle, since we are not distinguishing starting point, we will assume we begin the circuit by traversing the 4-cycle. There are 5*4*3*2 ways to choose the 4-cycle, and 4 possible starting points within the cycle, giving a total of 5*4*3*2/4 = 30 distinct possibilities. Once we have fixed these choices, note that the two 3-cycles must each consist of a diagonal of the 4-cycle plus two edges containing the point not included in the 4-cycle. Following the 4-cycle, the either the next vertex in the circuit is this fifth point, or it is the opposite diagonal and then the fifth point (2 choices). From the fifth point the 3-cycle which has not been traversed at all must be followed in one of 2 directions (2 choices), and then the final step back to the starting vertex is determined. This gives a total of 30*2*2 = **120** possible cycles. Note that if we reverse the order of the circuit, we do not get a circuit which starts with a 4-cycle so we don't need to correct for over-counting as we did above.*

Note that it was necessary to consider these two cycle decompositions separately in order to count them correctly because they have different symmetries. It is easy to get the wrong answer if you don't do this.

6. Consider the graph of H_n, the n-dimensional hypercube. The vertices of H_n can be labeled with binary strings of length n so that H_n contains 2^n vertices with edges between vertices which differ in 1 bit, e.g. the edges of H_2 are {00,01}, {01,11}, {11,10} and {10,00}. Note that H_n is a regular graph.

- What is the degree and diameter of H_n? ***n***
- For which values of n is H_n Eulerian? ***When n is even.***
- Find a Hamiltonian cycle for H_2, H_3, and H_4.

  H_2: 00, 01, 11, 10
  H_3: 000, 001, 011, 010, 110, 111, 101, 100
  H_4: 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100,
  1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000

- Prove that H_n is Hamiltonian for all n (hint: use induction on n).

  The base case is H_2 which is shown above. Given a Hamiltonian cycle for H_{n-1}, a Hamiltonian cycle for H_n can be constructed by using the cycle for H_{n-1} to traverse the vertices in H_n whose labels begin with 0, but instead of returning to the starting vertex traverse the vertices with labels beginning with 1 in reverse the order. See the Hamiltonian cycles for H_3 and H_4 above as examples.

7. Consider a set of 21 dominos with the ends of each domino labeled with between 1 and 6 dots and each combination uniquely represented. In class we proved that you cannot construct a cycle using all 21 dominos with adjacent edges matching. What is the largest legal domino cycle you can make with this set?

*If we remove 3 non-adjacent edges from K_6 all the vertices have even degree. Adding in the 6 self-loops doesn't change this, so the resulting graph contains an Eulerian circuit. Thus we can make a domino cycle with 21-3 = **18** dominos. This is the best possible, since in any other case the graph with edges corresponding to the dominos is not Eulerian.*

A standard set of dominoes contains 28 dominoes with the ends of each domino labeled with between 0 and 6 dots. Every combination is uniquely represented by one domino. Prove that you *can* construct a legal domino cycle using all the dominos in this set.

Every vertex in K_7 has even degree so it is Eulerian. This is still true when self-loops are added.

8. At an international peace conference two delegates from each of five countries are to be seated for dinner at a round table with 10 places. In order to promote international communication, the delegates should be seated so that the two delegates from the same country are not adjacent, and so that every possible combination of adjacent countries occurs around the table. Find a seating arrangement that satisfies these constraints.

Let A, B, C, D, and E represent the countries, and label the vertices of K_5 with these letters.. Using an Eulerian circuit of K_5 we can construct a seating *arrangement by writing down the vertices visited – each vertex will appear twice in our list, and we can put the two delegates for the country in the corresponding positions. In such a list the same letter can not appear consecutively, and every possible adjacent pair will occur exactly once.* ***ABCDECADBE*** *is one example.*

9. With a standard deck of 52 cards, what is the longest sequence of card you can construct so that any two adjacent cards are either the same suit or the same rank, but no three cards in a row have the same suit or rank?

A deck of cards can be represented by edges of the graph $K_{4,13}$*, since each card matches one of 4 suits with one of 13 ranks. This graph has 4 vertices with odd degree. Any sequence of cards satisfying the specified constraints corresponds to a trail in this graph. If we eliminate two non-adjacent edges, the resulting graph will have only 2 vertices with odd degree and thus will contain an Eulerian trail, but otherwise it will not. Thus 52-50 =* ***50*** *cards is the longest possible sequence.*