15.093 Optimization Methods

Lecture 11: Network Optimization
The Network Simplex Algorithm

Network Optimization

Why do we care?

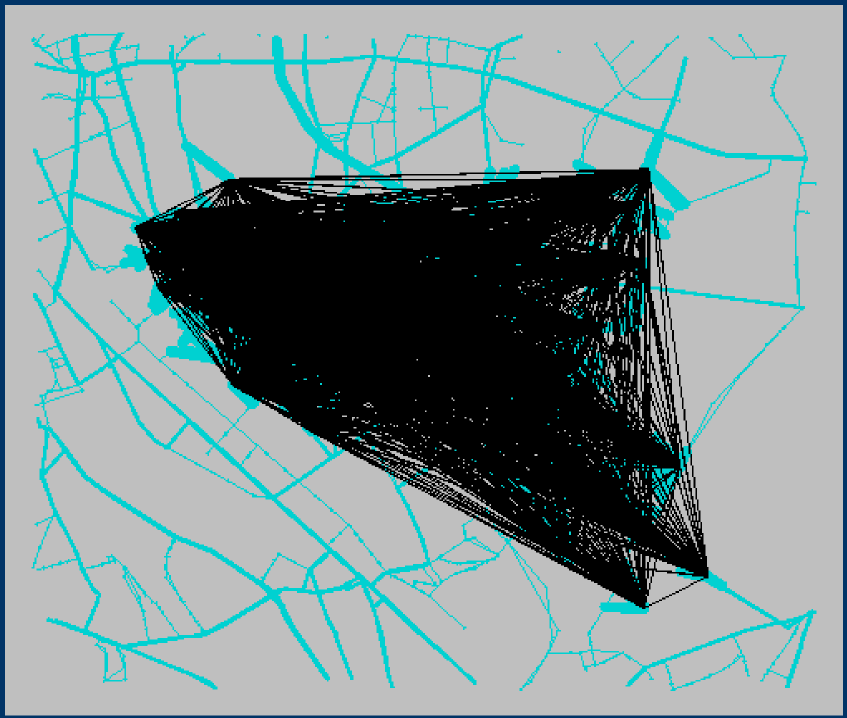
- Networks and associated optimization problems constitute reoccurring structures in many real-world applications.
- The network structure often leads to additional insight and improved understanding.
- Given integer data, the standard models have integer optimal solutions.
- The network structure also enables us to design more efficient algorithms.

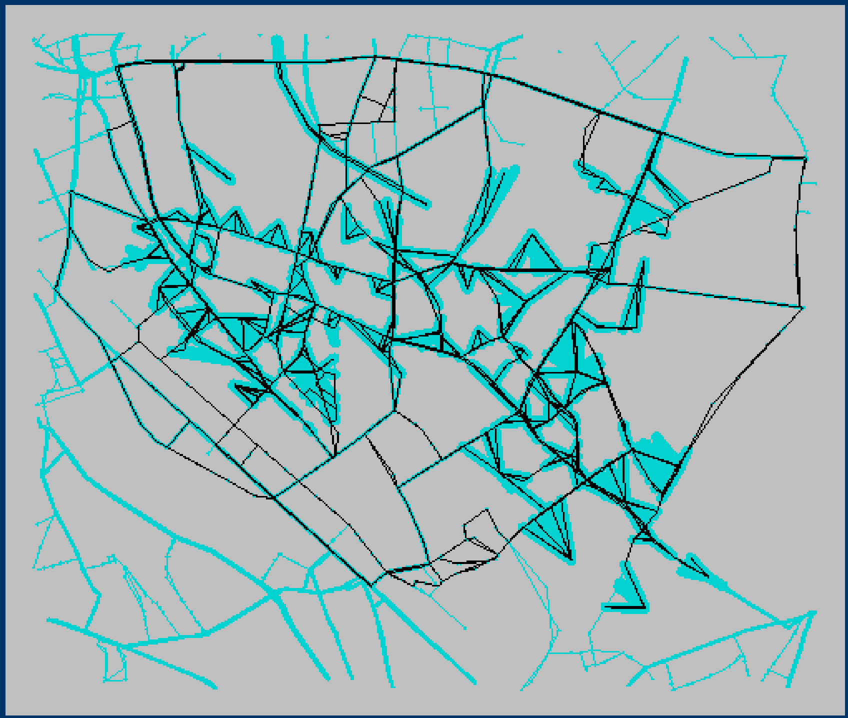
Network Optimization

A Comparison

Sample Instance...

$1,772$ nodes and $2,880$ arcs





A Comparison

Running Times

Algorithm	Running Time (sec)	# Iterations
Standard Simplex	334.59	42759
Network Simplex	7.37	23306
Ratio	2.2 %	54 %

Average over **5** random instances with **10,000** nodes and **25,000** arcs each.

Outline

Today's Lecture

- The Simplex Algorithm: A Reminder
- The Network Simplex: A Combinatorial View
- The Network Simplex: An Animated View
- The Network Simplex: An Algebraic View

The Simplex Algorithm

A Reminder

The Problem...

$$\begin{aligned} \min \;& c'x \\ \text{s.t.} \;& Ax = b \\ & x \geq 0 \end{aligned}$$

The Simplex Algorithm

A Reminder

The Algorithm

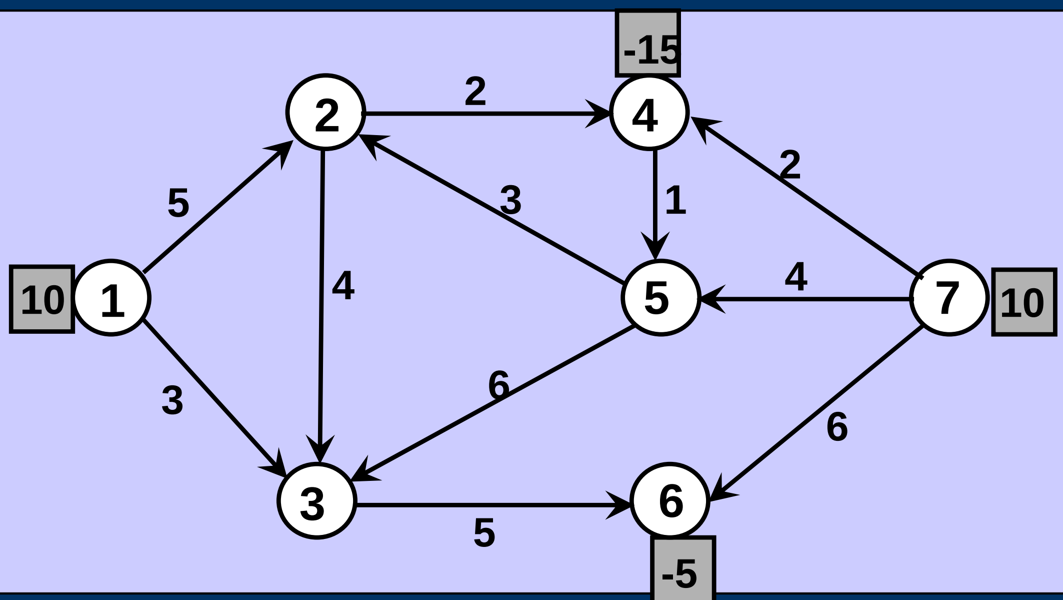
1. Start with basis $B = [A_{B(1)}, \ldots, A_{B(m)}]$ and BFS x.
2. Compute $\overline{c}_j = c_j - c'_B B^{-1} A_j$.
   - If $\overline{c}_j \geq 0$; x optimal; stop.
   - Select j such that $\overline{c}_j < 0$.
3. Compute $u = B^{-1} A_j$. $\theta^* = \min_{1 \leq i \leq m, u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$.
4. Form a new basis by replacing $A_{B(\ell)}$ with A_j.
5. $y_j = \theta^*$; $y_{B(i)} = x_{B(i)} - \theta^* u_i$.

The Network Simplex Algorithm

The Problem

Combinatorially...

Determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes. Arcs have costs associated with them.



The Network Simplex Algorithm

The Problem

Algebraically...

- Network $G = (N, A)$.
- Arc costs $c : A \to \mathbb{Z}$.
- Node balances $b : N \to \mathbb{Z}$.

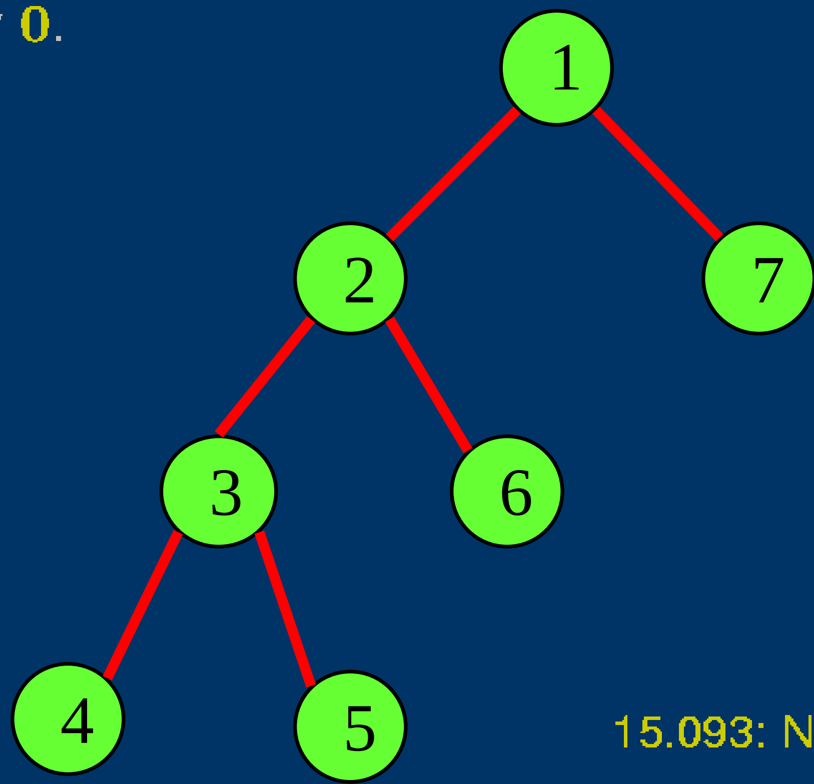
$$
\begin{array}{lrl}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i & \text{for all } i \in N \\
& x_{ij} \geq 0 & \text{for all } (i,j) \in A
\end{array}
$$

The Network Simplex Algorithm

Tree Solutions

Definition...

- A *tree* is a graph that is connected and has no cycles.
- A *spanning tree* of a graph G is a subgraph that is a tree and contains all nodes of G.
- A flow x forms a *tree solution* with a spanning tree of the network if every non-tree arc has flow 0.



The Network Simplex Algorithm

Tree Solutions

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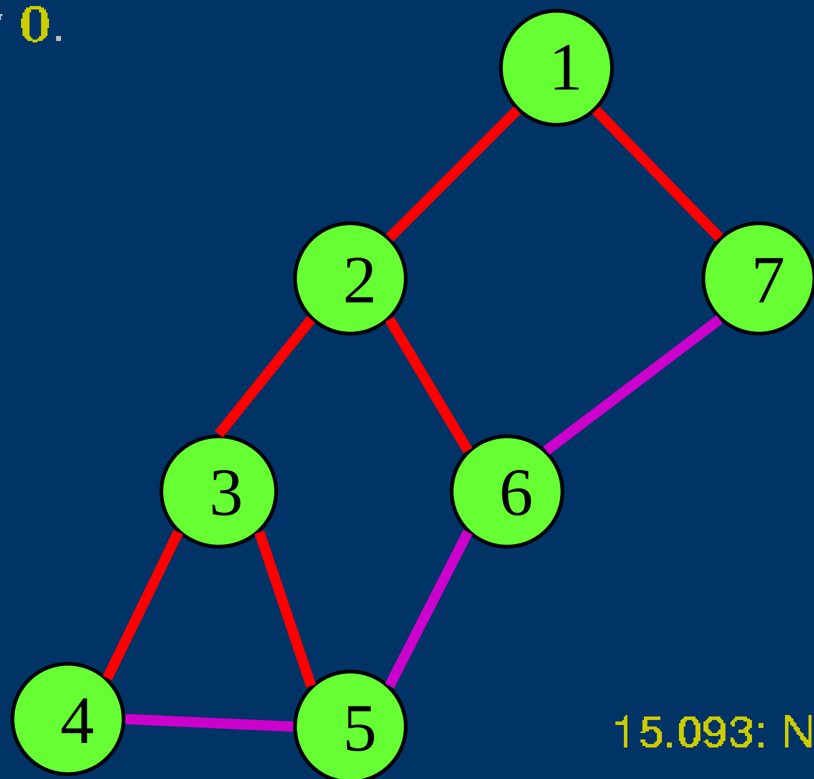

The Network Simplex Algorithm

Tree Solutions

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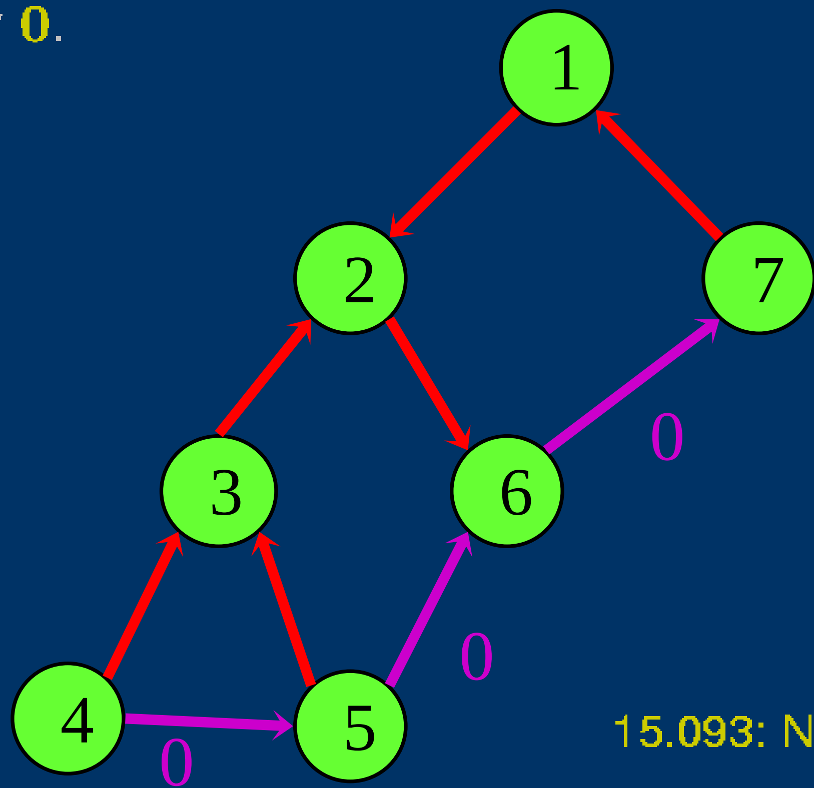
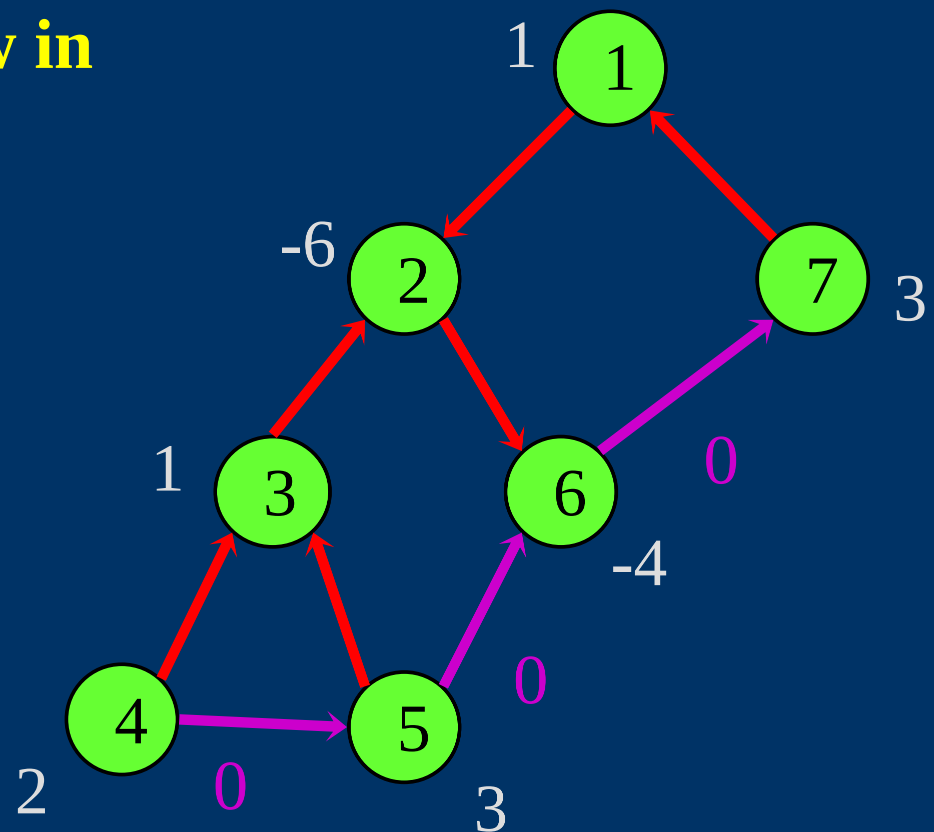


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (4,3)?

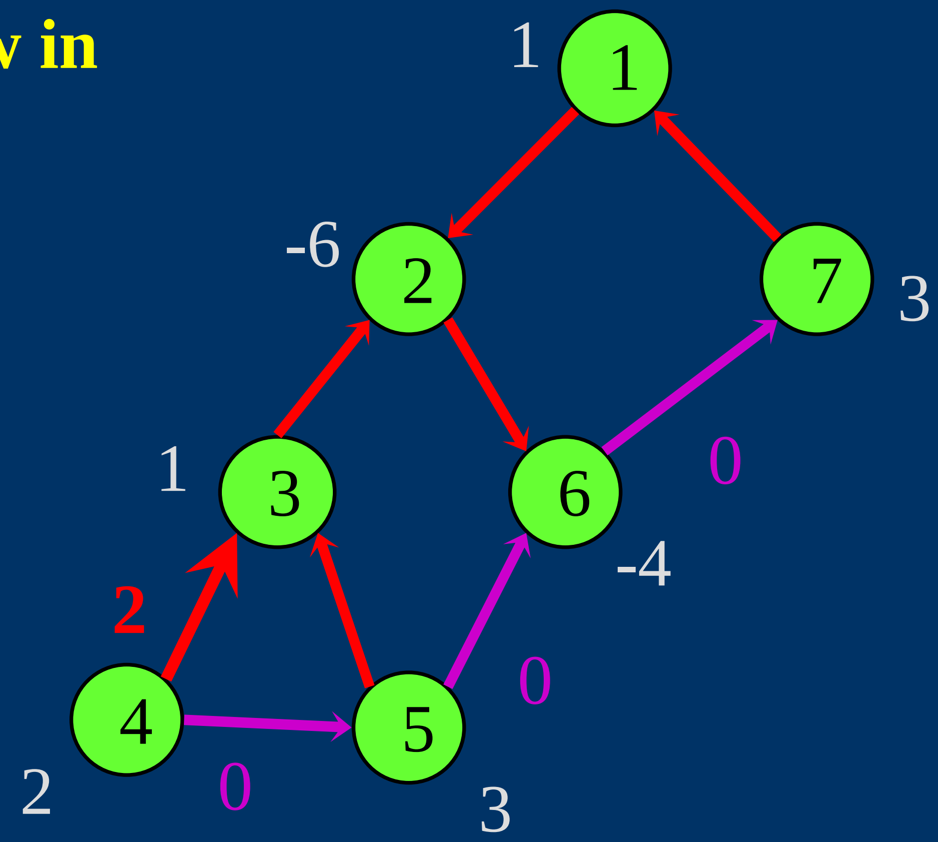


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (5,3)?

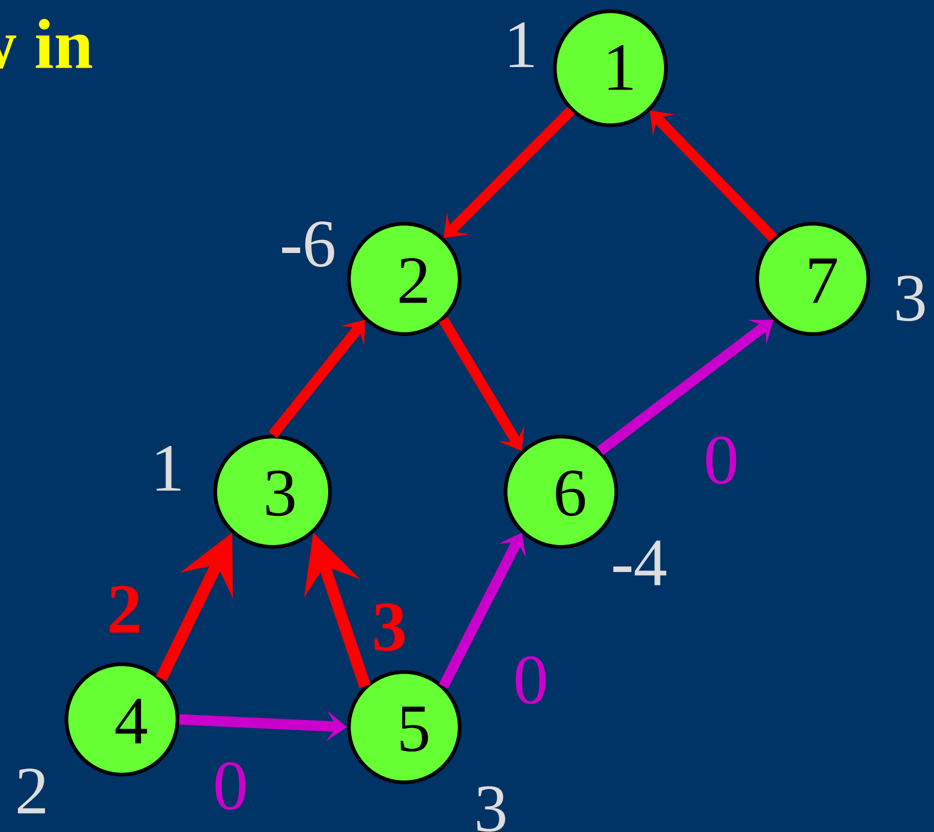


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (3,2)?

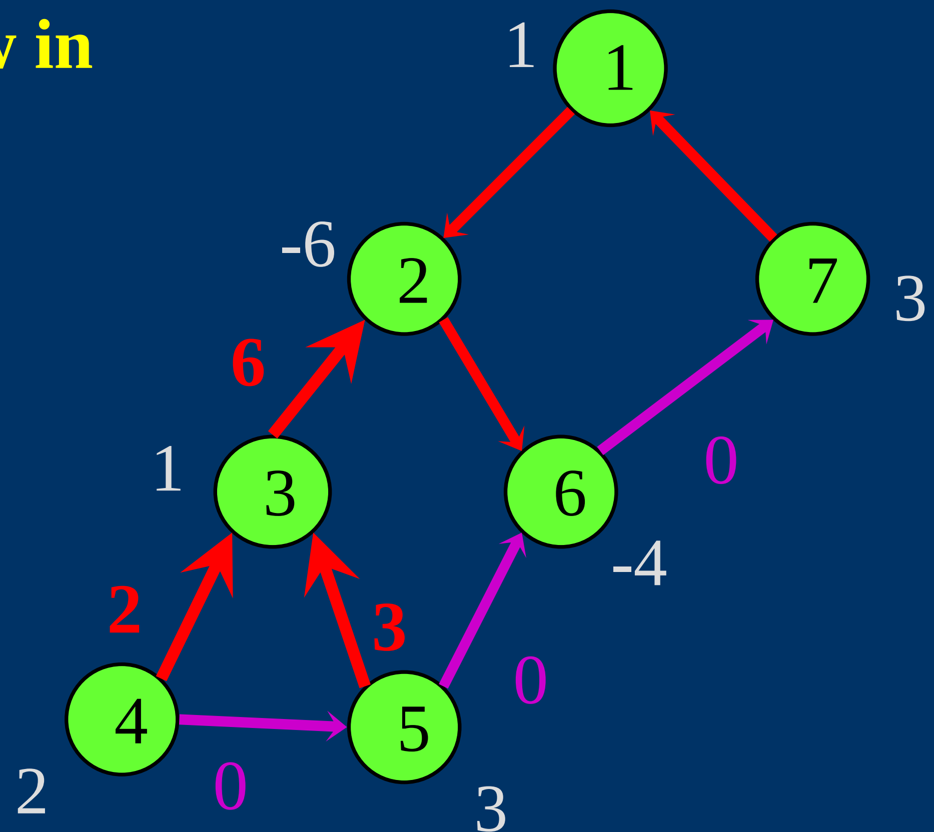


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (2,6)?

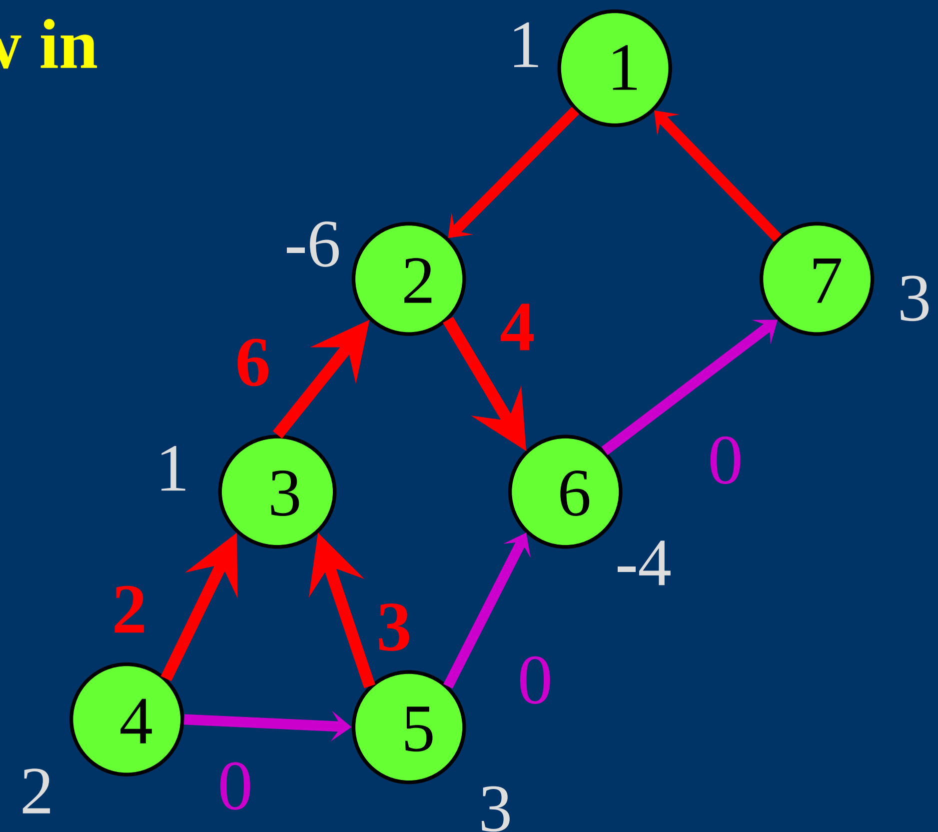


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (7,1)?

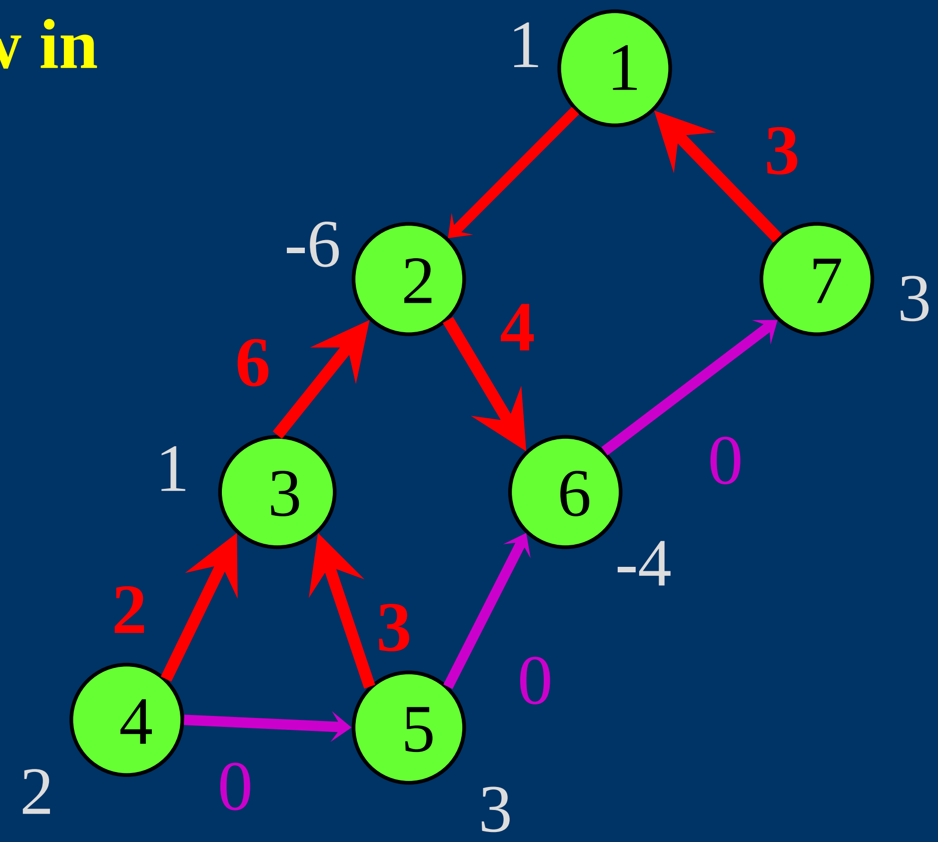


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

What is the flow in arc (1,2)?

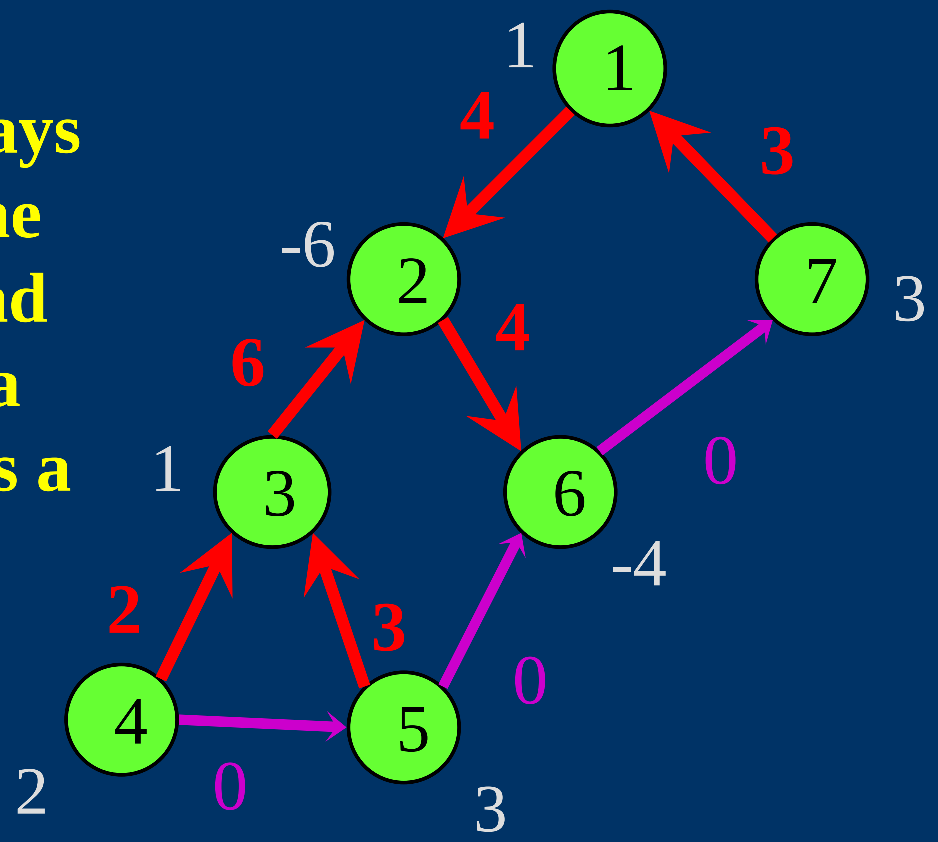


The Network Simplex Algorithm

Tree Solutions

Computing the Flow...

Note: there are two different ways of calculating the flow on (1,2), and both ways give a flow of 4. Is this a coincidence?



Tree Solutions

Trees vs. Tree Flows...

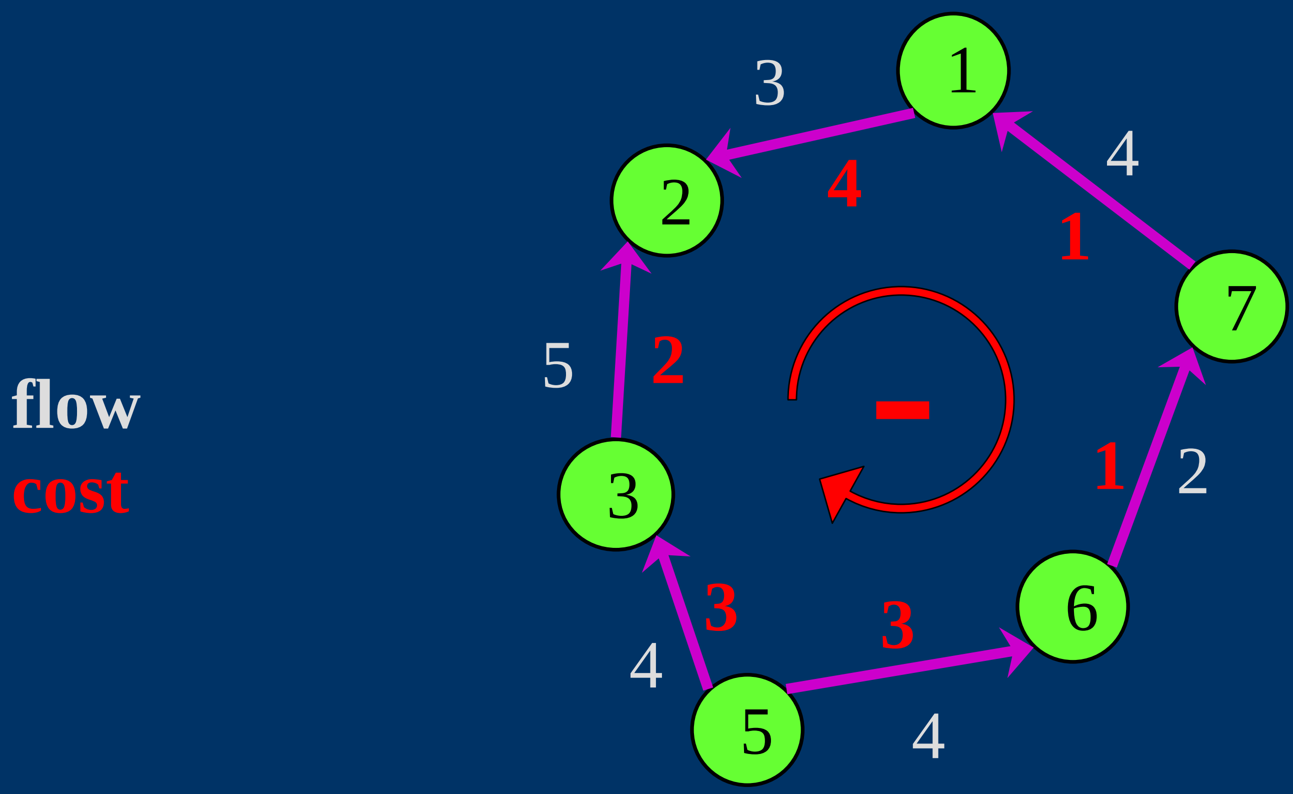
- Every tree flow has a corresponding tree (and perhaps more than one).
- Given a tree, we obtain a unique tree flow associated with it.

The Network Simplex Algorithm

Tree Solutions

BFS Property...

Theorem 1 *If the objective function is bounded from below, a min-cost flow problem always has an optimal tree solution.*



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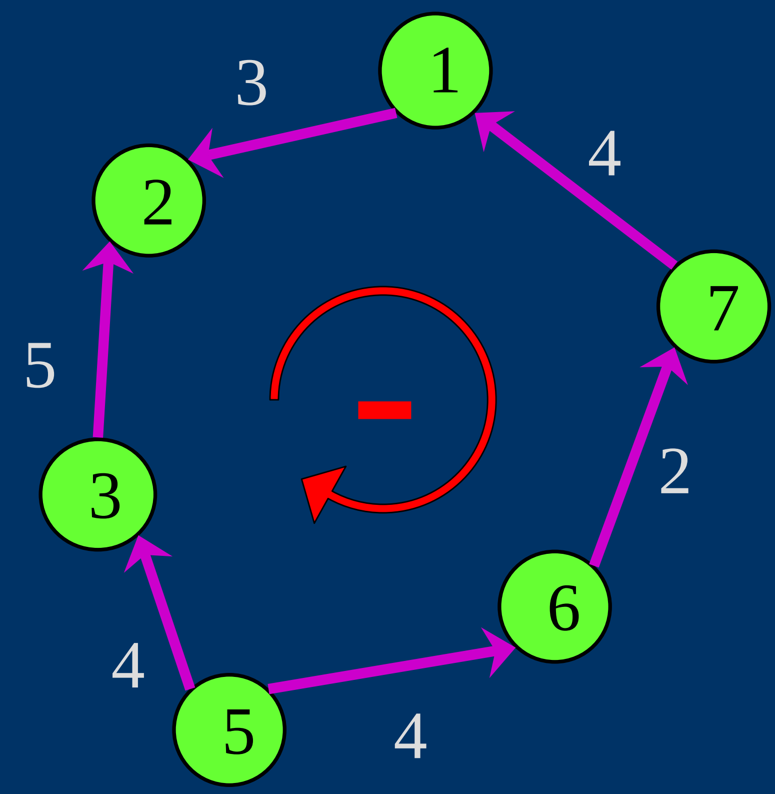

The Network Simplex Algorithm

Tree Solutions

BFS Property...

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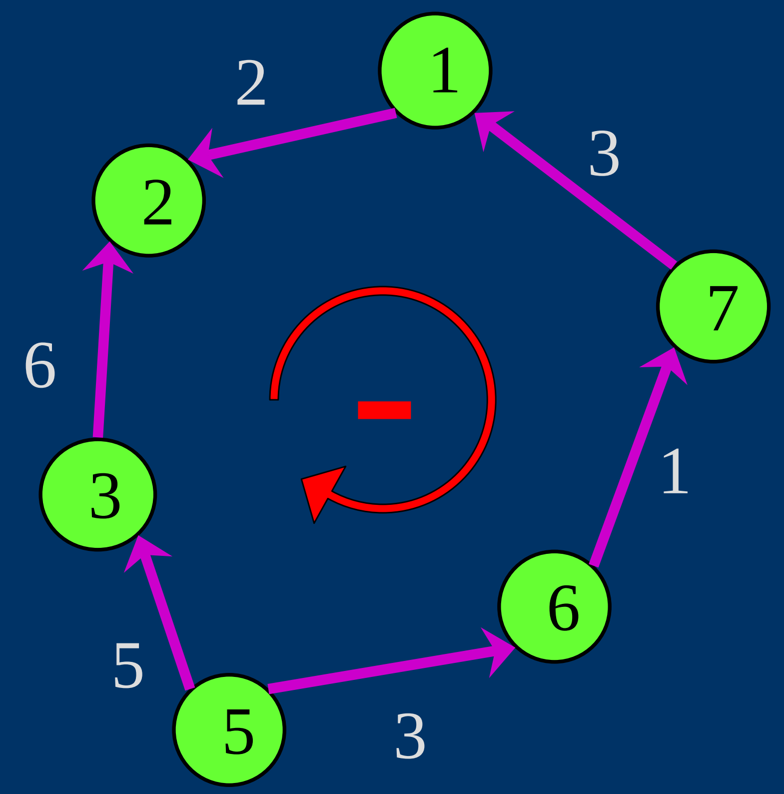

The Network Simplex Algorithm

Tree Solutions

BFS Property...

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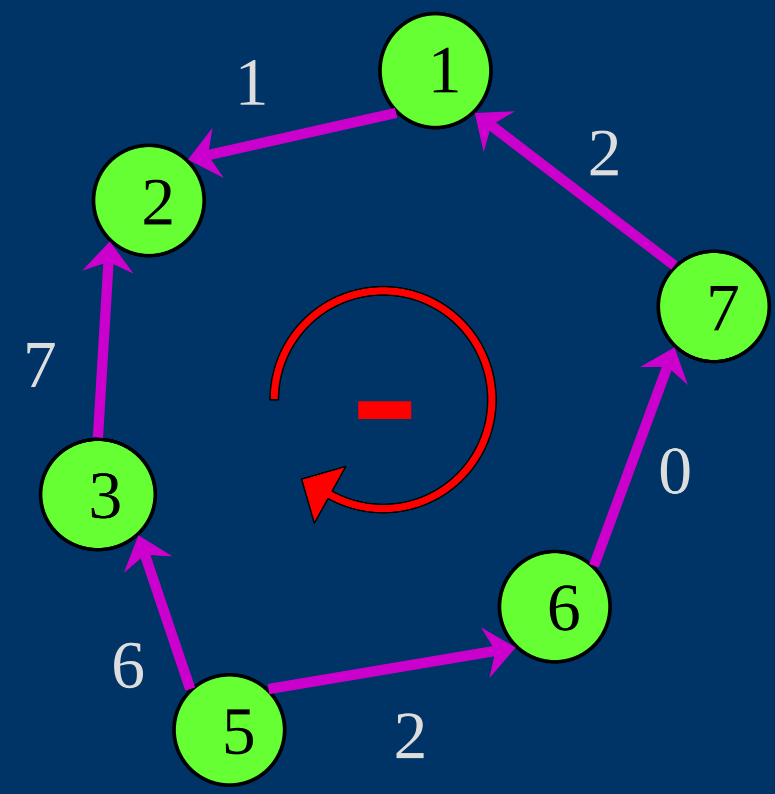

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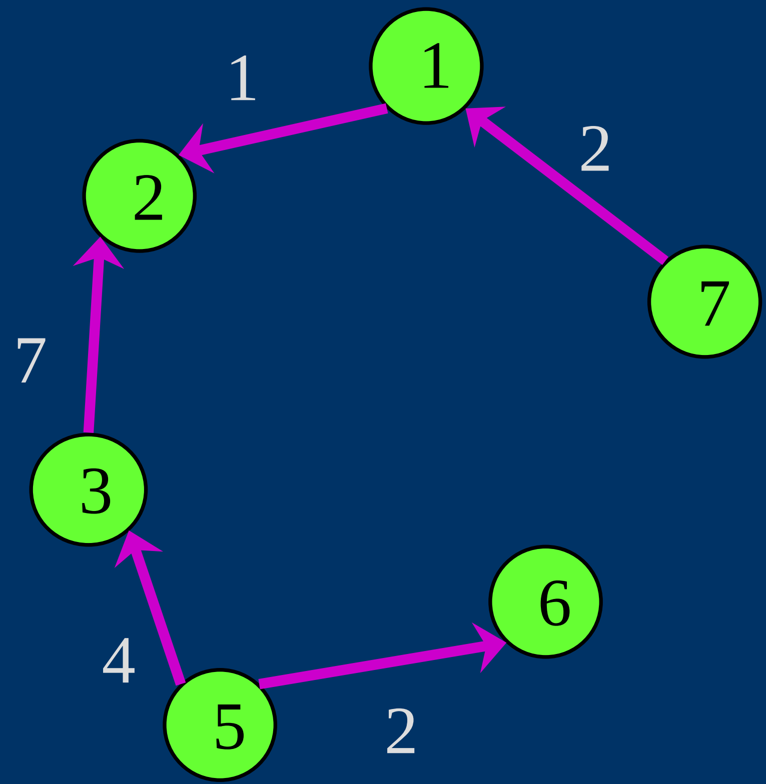


The Network Simplex Algorithm

Tree Solutions

Optimality Condition...

Theorem 2 *A (feasible) tree T is optimal if, for some choice of node potentials p_i,*

(a) $\bar{c}_{ij} = c_{ij} - p_i + p_j = 0$ *for all* $(i,j) \in T$*,*

(b) $\bar{c}_{ij} = c_{ij} - p_i + p_j \geq 0$ *for all* $(i,j) \in A \setminus T$*.*

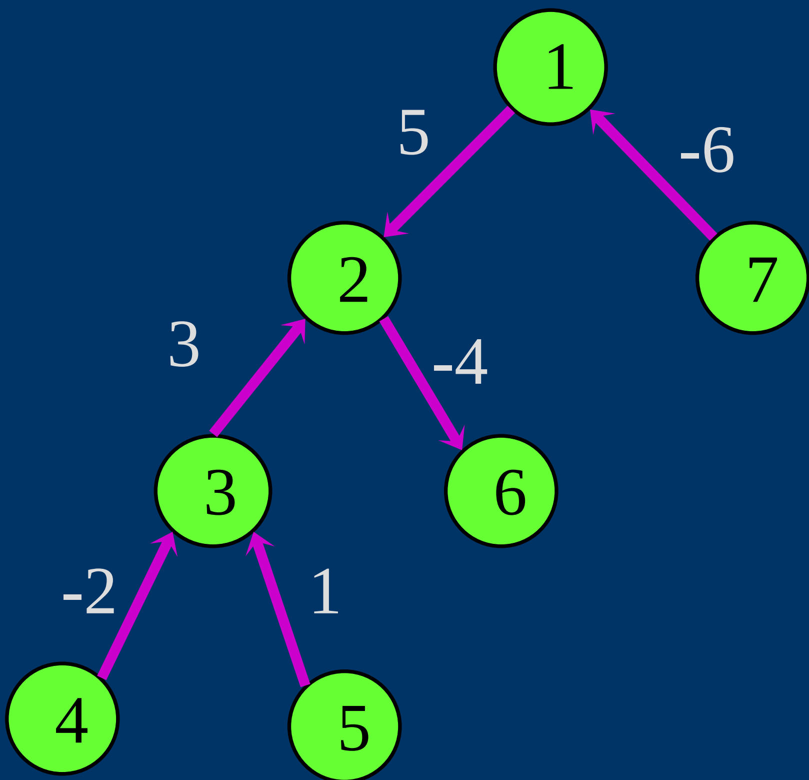
Proof:

- $\min \sum_{(i,j)\in A} c_{ij}x_{ij}$ is equivalent to $\min \sum_{(i,j)\in A} \bar{c}_{ij}x_{ij}$.
- $\min \sum_{(i,j)\in A} \bar{c}_{ij}x_{ij}$ is equivalent to $\min \sum_{(i,j)\in A\setminus T} \bar{c}_{ij}x_{ij}$.
- For any solution x, $x_{ij} \geq x^*_{ij}$ for all $(i,j) \in A \setminus T$.

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

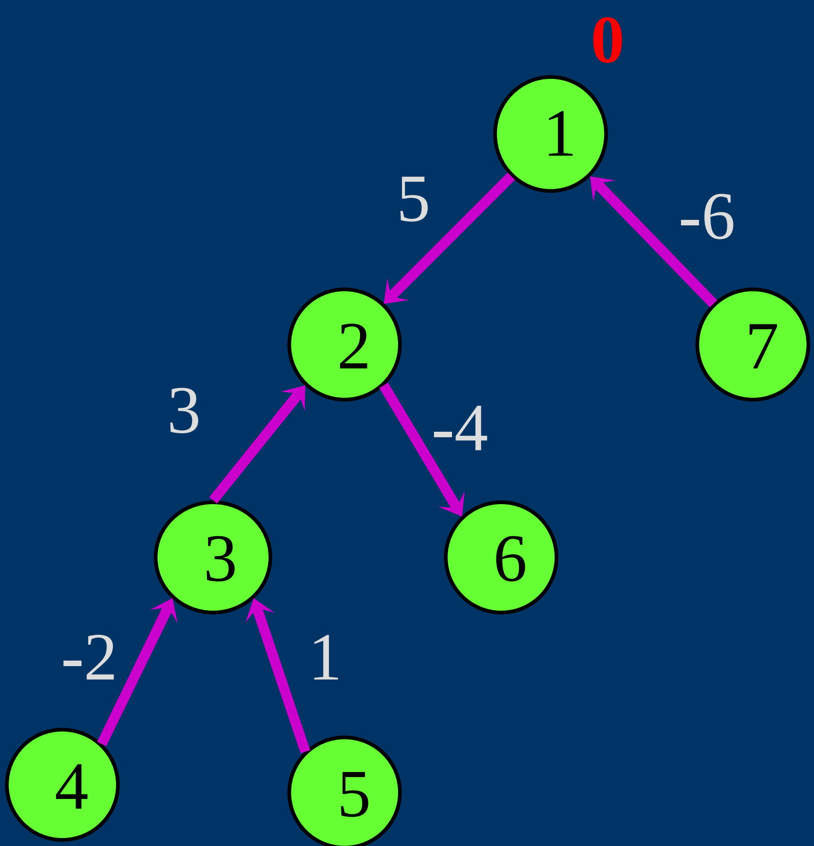


Here is a spanning tree with arc costs. How can one choose node potentials so that reduced costs of tree arcs are 0?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...



There is a redundant constraint in the minimum cost flow problem.

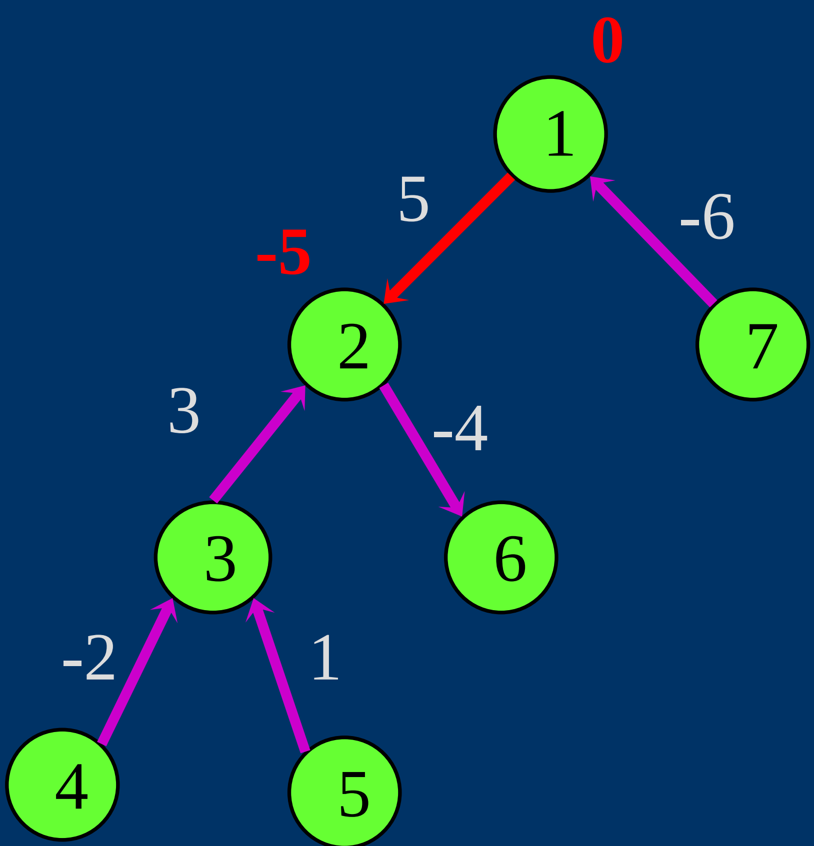
One can set p_1 arbitrarily. We will let $p_1 = 0$.

What is the node potential for 2?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

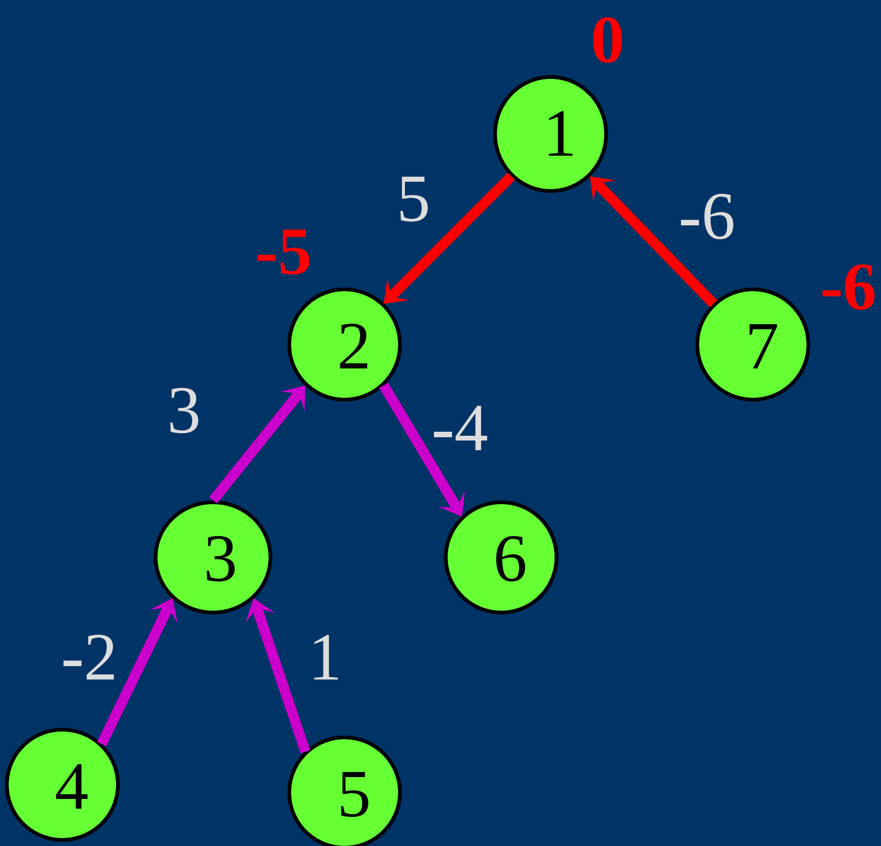


What is the node potential for 7?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

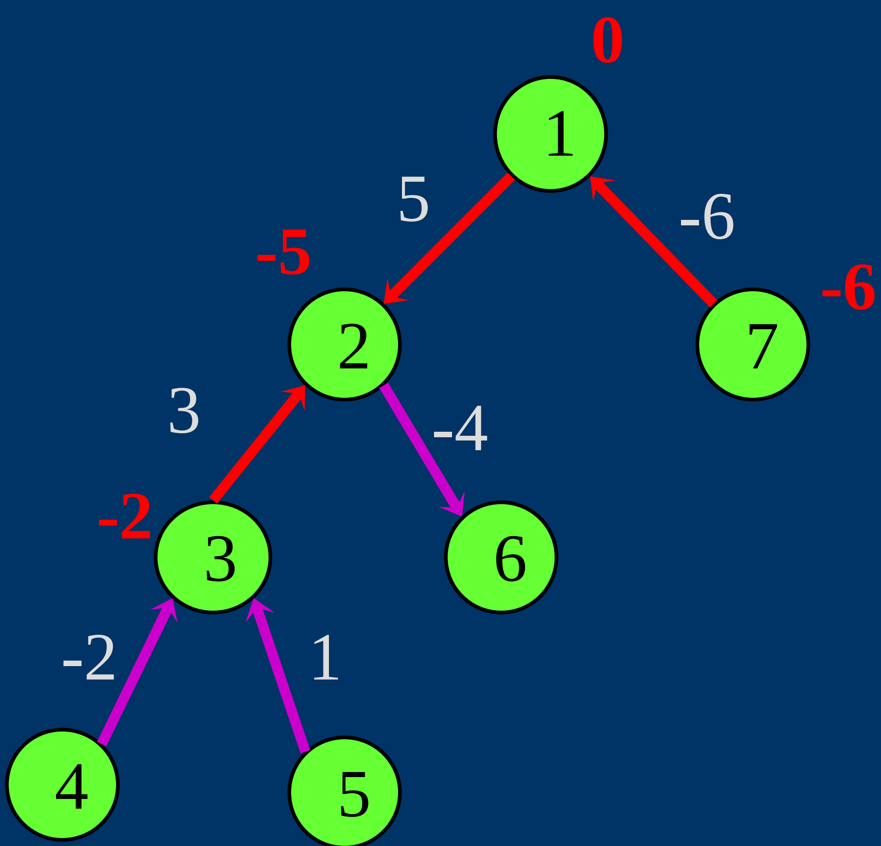


What is the potential for node 3?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

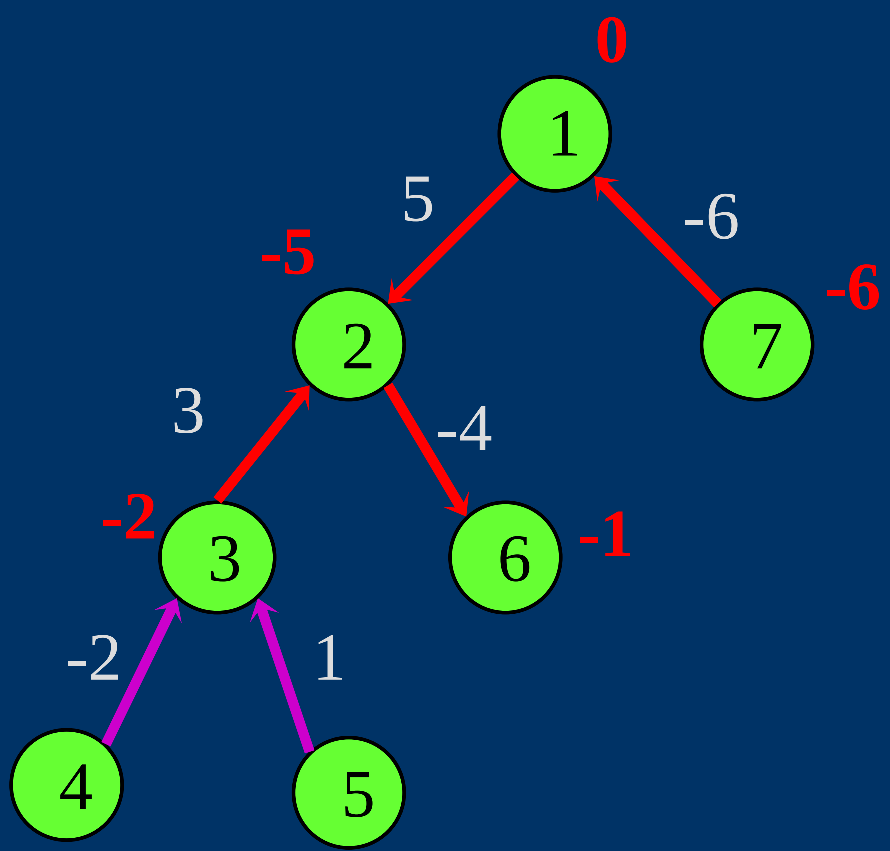


What is the potential for node 6?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

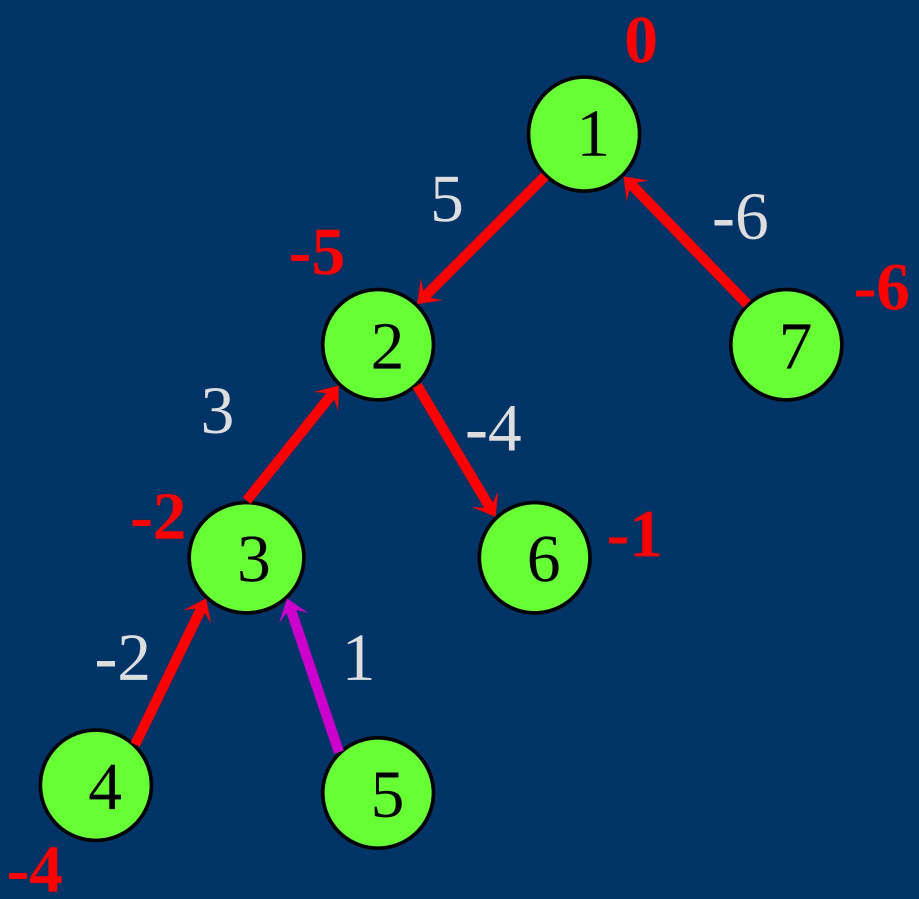


What is the potential for node 4?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...

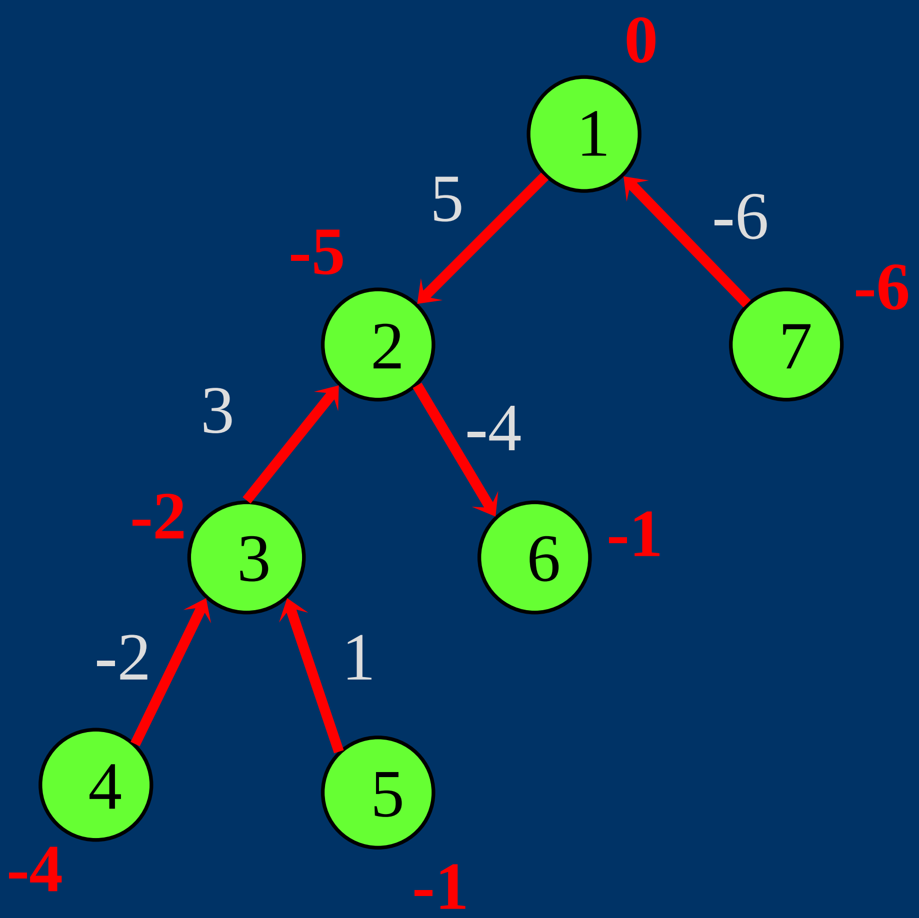


What is the potential for node 5?

The Network Simplex Algorithm

Tree Solutions

Computing Node Potentials...



These are the node potentials associated with this tree. They do not depend on arc flows, nor on costs of non-tree arcs.

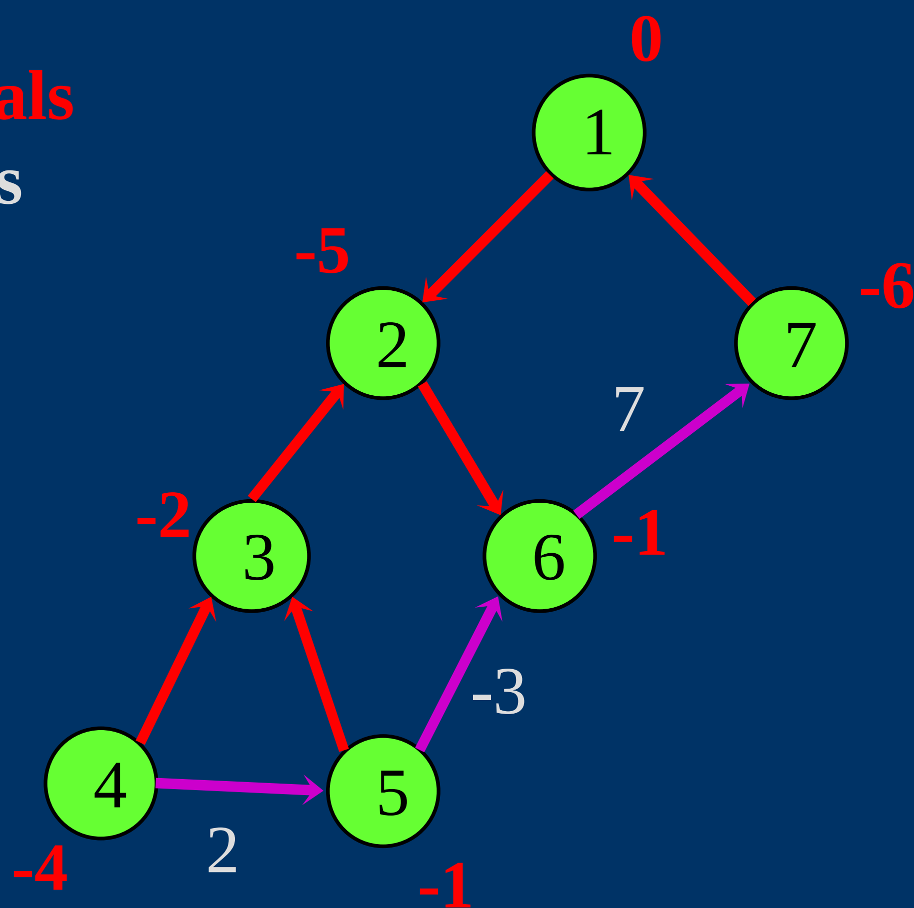
The Network Simplex Algorithm

Tree Solutions

Updating the Tree...

Node potentials

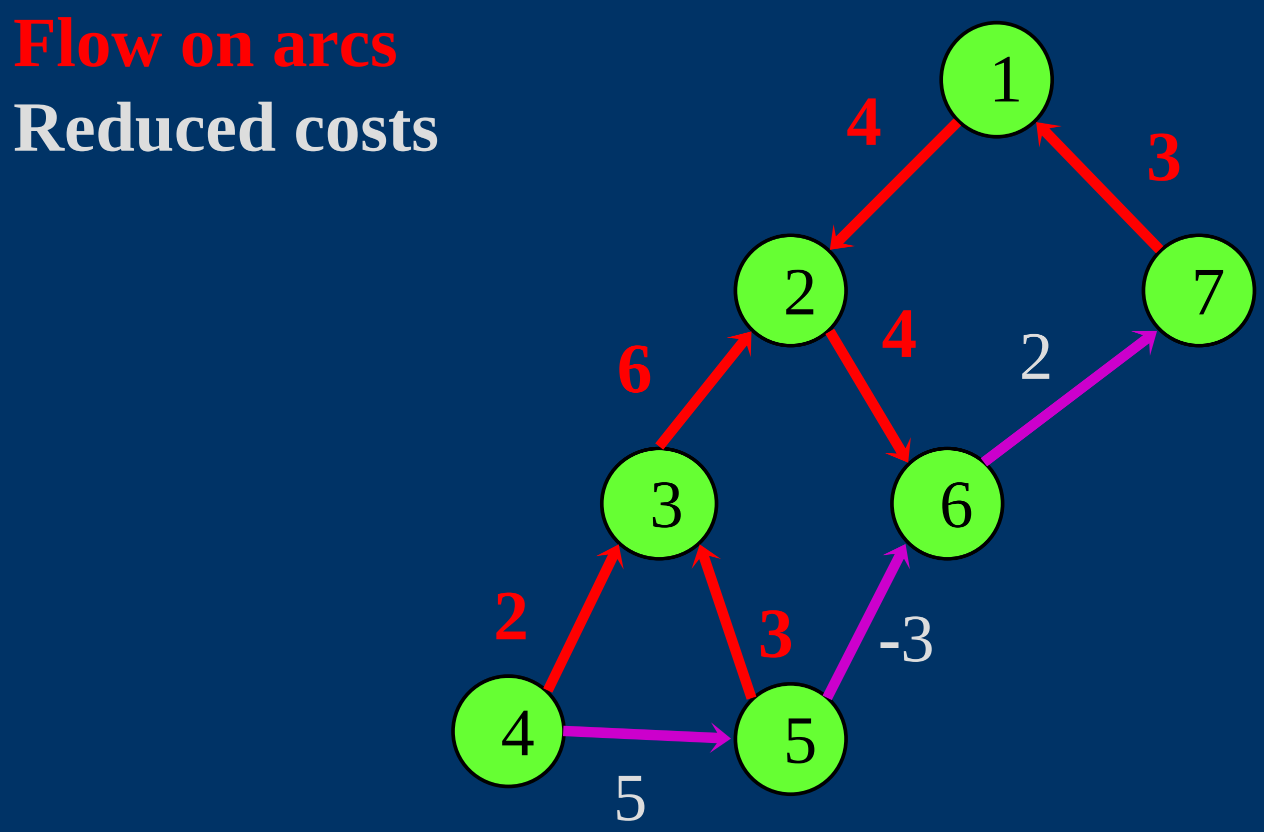
Original costs



The Network Simplex Algorithm

Tree Solutions

Updating the Tree...

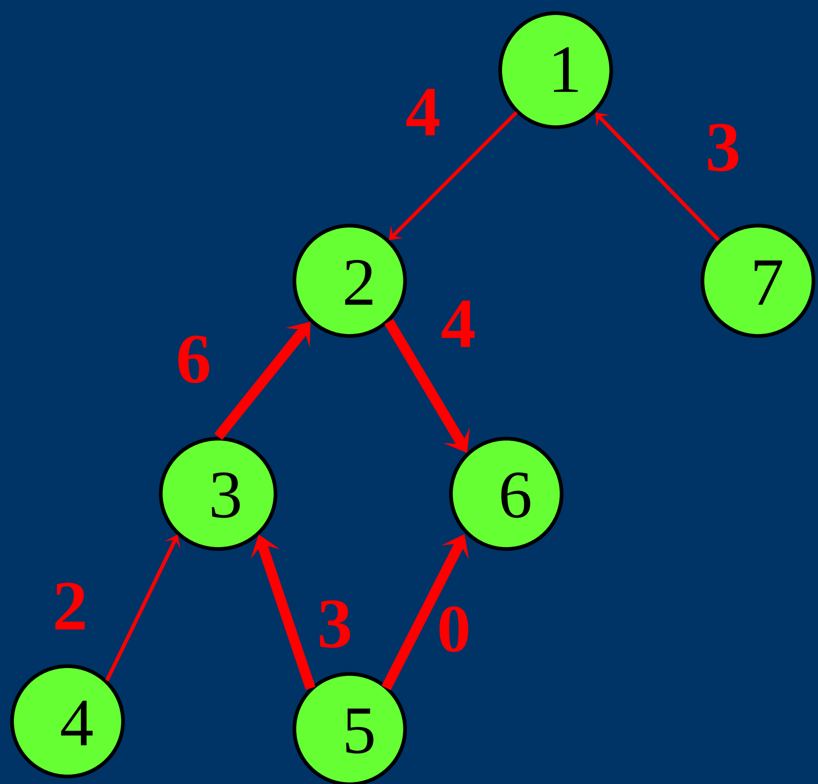


The Network Simplex Algorithm

Tree Solutions

Updating the Tree...

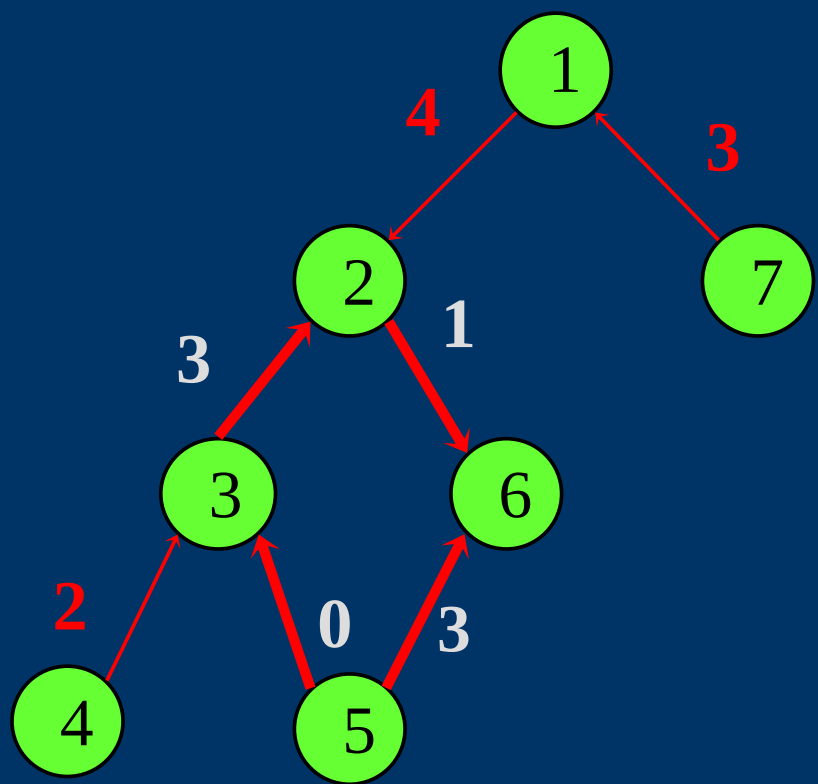
Flow on arcs



The Network Simplex Algorithm

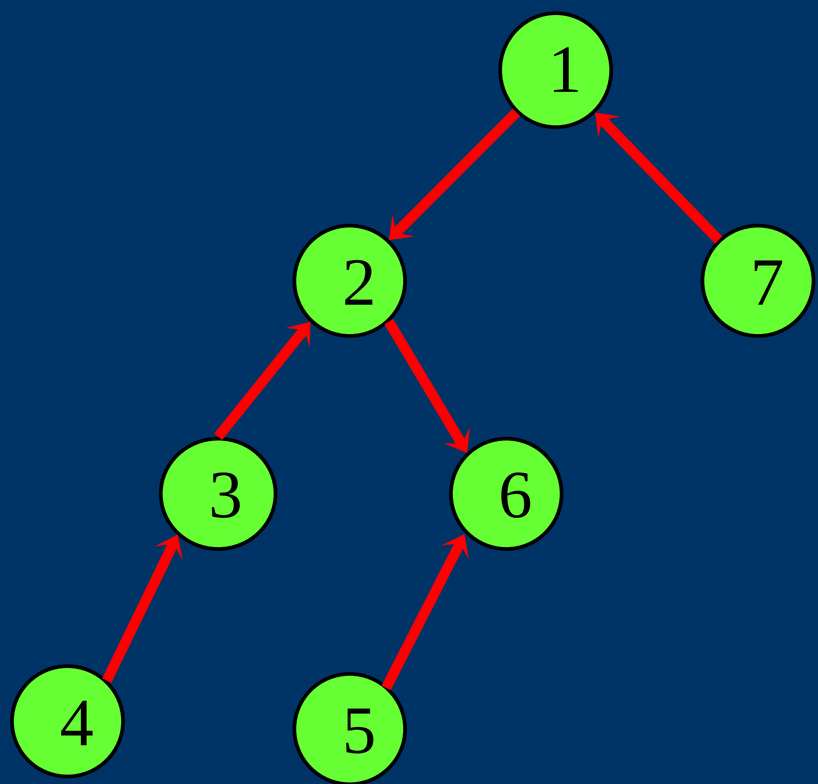
Tree Solutions

Updating the Tree...



Tree Solutions

Updating the Tree...



Overview of the Algorithm

1. Determine an initial feasible tree $\boldsymbol{T}$. Compute flow $\boldsymbol{x}$ and node potentials $\boldsymbol{p}$ associated with $\boldsymbol{T}$.
2. Calculate $\overline{c}_{ij} = c_{ij} - p_i + p_j$ for $(i,j) \notin T$.
   - If $\overline{c} \geq 0$, x optimal; stop.
   - Select (i,j) with $\overline{c}_{ij} < 0$.
3. Add (i,j) to T creating a unique cycle C. Send a maximum flow around C while maintaining feasibility. Suppose the exiting arc is (k,ℓ).
4. $T := (T \setminus (k,\ell)) \cup (i,j)$.

Min-Cost Flow

Integrality

Our reasoning has two important and far-reaching implications:

- There always exists an integer optimal flow (if node balances b_i are integer).
- There always exist optimal integer node potentials (if arc costs c_{ij} are integer).

The Network Simplex Algorithm

An Animation

The Algebraic View

- Bases and trees.
- Dual variables and node potentials.
- Changing bases and updating trees.
- Optimality testing.

The Network Simplex Algorithm

The Algebraic View

Bases vs. Trees...

The constraint matrix A of the min-cost flow problem is the node-arc incidence matrix of the underlying network.

	(1, 2)	(2, 6)	(3, 2)	(4, 3)	(4, 5)	(5, 3)	(5, 6)	(6, 7)	(7, 1)
1	+1	0	0	0	0	0	0	0	−1
2	−1	+1	−1	0	0	0	0	0	0
3	0	0	+1	−1	0	−1	0	0	0
4	0	0	0	+1	+1	0	0	0	0
5	0	0	0	0	−1	+1	+1	0	0
6	0	−1	0	0	0	0	−1	+1	0
7	0	0	0	0	0	0	0	−1	+1

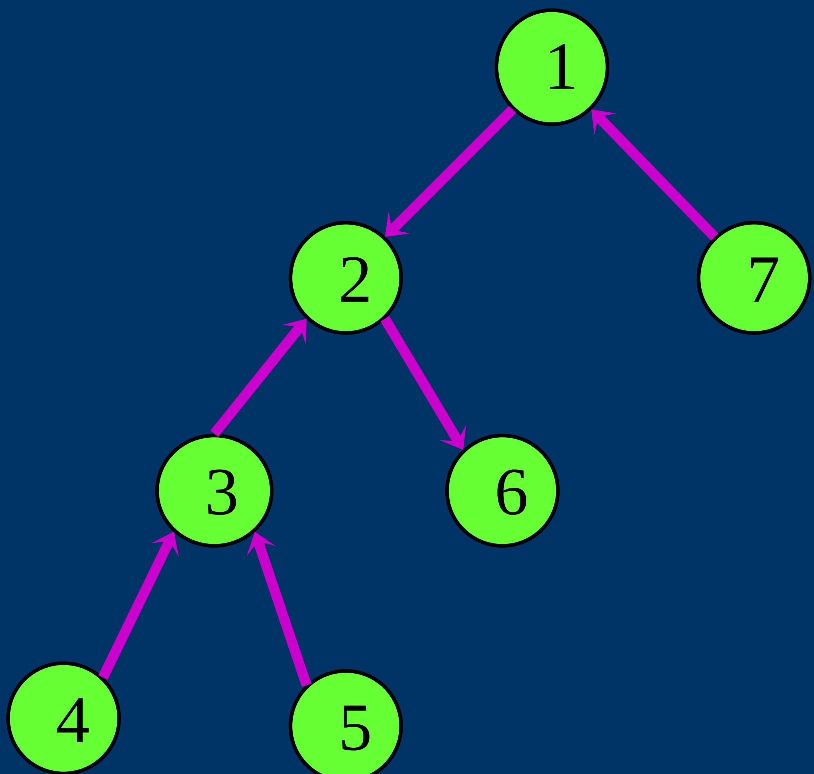
The rows of A are linearly dependent.

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Let B be the submatrix corresponding to the tree



	(1, 2)	(2, 6)	(3, 2)	(4, 3)	(5, 3)	(7, 1)
1	+1	0	0	0	0	−1
2	−1	+1	−1	0	0	0
3	0	0	+1	−1	−1	0
4	0	0	0	+1	0	0
5	0	0	0	0	+1	0
6	0	−1	0	0	0	0
7	0	0	0	0	0	+1

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

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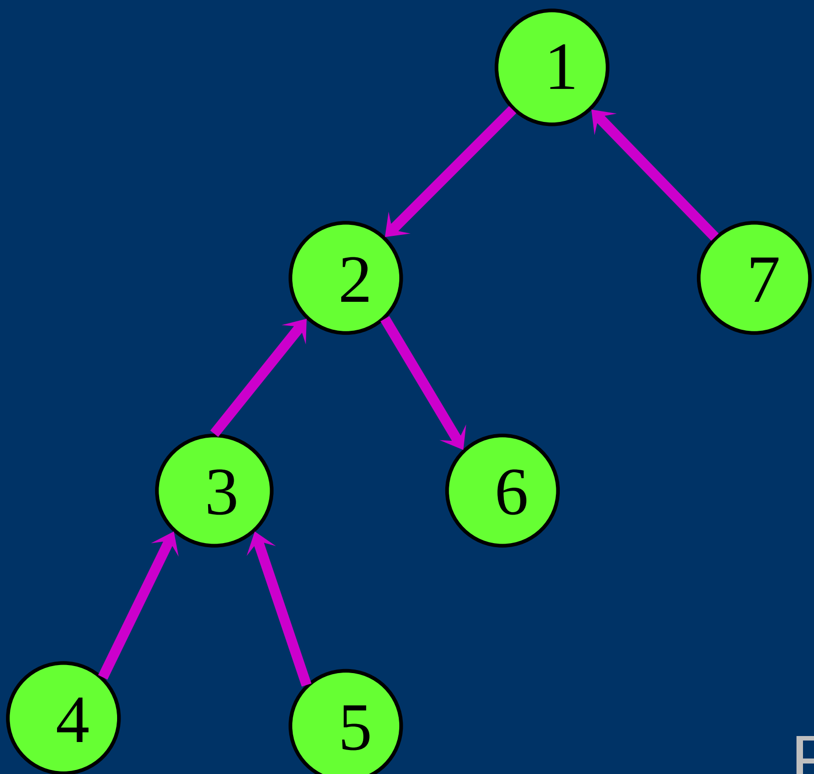


	(1, 2)	(2, 6)	(3, 2)	(4, 3)	(5, 3)	(7, 1)
4	0	0	0	+1	0	0
5	0	0	0	0	+1	0
6	0	−1	0	0	0	0
7	0	0	0	0	0	+1
3	0	0	+1	−1	−1	0
2	−1	+1	−1	0	0	0
1	+1	0	0	0	0	−1

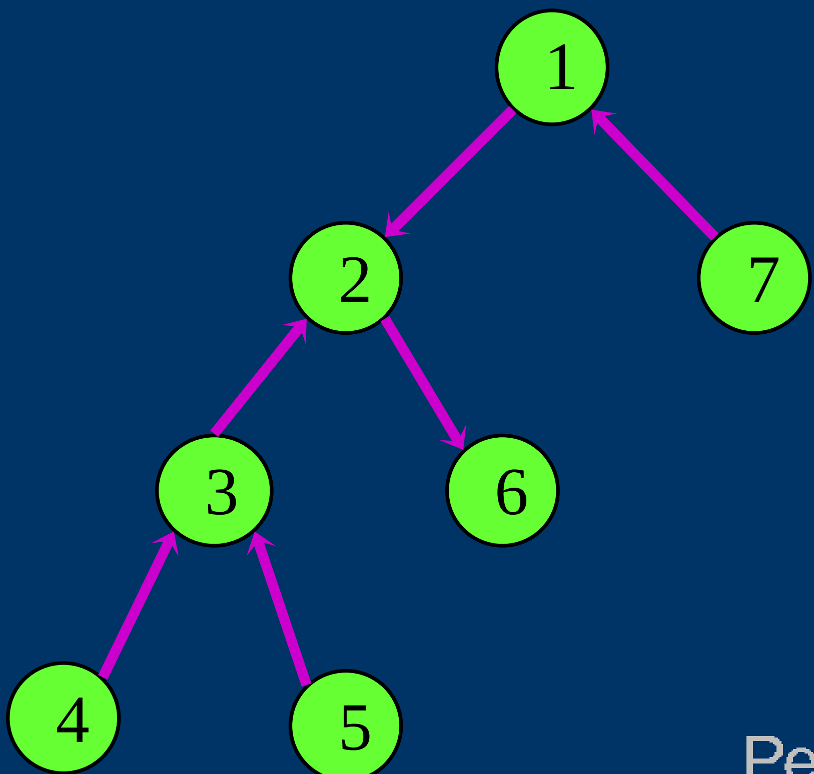
Permuting Rows

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Let B be the submatrix corresponding to the tree



	(4,3)	(5,3)	(2,6)	(7,1)	(3,2)	(1,2)
4	+1	0	0	0	0	0
5	0	+1	0	0	0	0
6	0	0	−1	0	0	0
7	0	0	0	+1	0	0
3	−1	−1	0	0	+1	0
2	0	0	+1	0	−1	−1
1	0	0	0	−1	0	+1

Permuting Columns

Corollary 1

(a) *The matrix* A *has rank* $n - 1$.

(b) *Every tree solution is a basic solution.*

The Network Simplex Algorithm

The Algebraic View

...Bases vs. Trees...

Theorem 3 *Every tree defines a basis and, conversely, every basis definies a tree.*

Suppose the graph defined by a basis contains a cycle $1 - 2 - 3 - 4 - 5 - 6$:

	(1, 2)	(2, 3)	(4, 3)	(5, 4)	(5, 6)	(1, 6)
1	+1	0	0	0	0	+1
2	−1	+1	0	0	0	0
3	0	−1	−1	0	0	0
4	0	0	+1	−1	0	0
5	0	0	0	+1	+1	0
6	0	0	0	0	−1	−1

The Network Simplex Algorithm

The Algebraic View

Dual Variables vs. Node Potentials...

Remember, the simplex algorithm computes the dual variables $\boldsymbol{p}$ as the solution to $\boldsymbol{p}'\boldsymbol{B} = \boldsymbol{c}'_{\boldsymbol{B}}$.

$$(p_4, p_5, p_6, p_7, p_3, p_2) \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ -1 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & -1 & -1 \end{pmatrix}$$

$$= (c_{43},\ c_{53},\ c_{26},\ c_{71},\ c_{32},\ c_{12})$$

Hence, $p_2 = -c_{12}$, $p_3 = c_{32} + p_2$, $p_7 = c_{71}$, ...

The Network Simplex Algorithm

The Algebraic View

Optimality Testing...

Remember, the simplex algorithm computes the reduced costs $\bar{c}$ as $\bar{c}_{ij} = c_{ij} - p' A_{ij}$.

	(1, 2)	(2, 6)	(3, 2)	(4, 3)	(4, 5)	(5, 3)	(5, 6)	(6, 7)	(7, 1)
1	+1	0	0	0	0	0	0	0	−1
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3	0	0	+1	−1	0	−1	0	0	0
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5	0	0	0	0	−1	+1	+1	0	0
6	0	−1	0	0	0	0	−1	+1	0
7	0	0	0	0	0	0	0	−1	+1

Therefore, $\bar{c}_{ij} = c_{ij} - p_i + p_j$.

Summary

- The network simplex algorithm is extremely fast in practice.
- Relying on network data structures, rather than matrix algebra, causes the speedups. It leads to simple rules for selecting the entering and exiting variables.
- Running time per pivot:
  - arcs scanned to identify an entering arc,
  - arcs scanned of the basic cycle,
  - nodes of the subtree.
- A good pivot rule can dramatically reduce running time in practice.

15.093J / 6.255J Optimization Methods
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