15.093: Optimization Methods

Lecture 9: Large Scale Optimization

1 Outline

SLIDE 1

1. The idea of column generation
2. The cutting stock problem
3. Stochastic programming

2 Column Generation

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- For $\boldsymbol{x} \in \Re^n$ and n large consider the LOP:

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Restricted problem

$$\begin{aligned} \min \quad & \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad & \sum_{i \in I} \boldsymbol{A_i} x_i = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

2.1 Two Key Ideas

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- Generate columns $\boldsymbol{A_j}$ only as needed.
- Calculate $\min_i \overline{c}_i$ efficiently without enumerating all columns.

3 The Cutting Stock Problem

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- Company has a supply of large rolls of paper of width W.
- b_i rolls of width $w_i,\ i = 1, \ldots, m$ need to be produced.
- Example: $w = 70$ inches, can be cut in 3 rolls of width $w_1 = 17$ and 1 roll of width $w_2 = 15$, waste:

$$70 - (3 \times 17 + 1 \times 15) = 4$$

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- Given $w_1, \ldots, w_m$ and W there are many cutting patterns: $(3, 1)$ and $(2, 2)$ for example

$$\begin{aligned} 3 \times 17 + 1 \times 15 & \leq 70 \\ 2 \times 17 + 2 \times 15 & \leq 70 \end{aligned}$$

- Pattern: $(a_1, \dots, a_m)$ integers:

$$\sum_{i=1}^{m} a_i w_i \le W$$

3.1 Problem

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- Given $w_i,\ b_i,\ i = 1, \dots, m$ (b_i: number of rolls of width w_i demanded, and W (width of large rolls):
- Find how to cut the large rolls in order to minimize the number of rolls used.

3.2 Concrete Example

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- What is the solution for $W = 70, w_1 = 21, w_2 = 11, b_1 = 40, b_2 = 40$?
- feasible patterns: $(2, 2)$, $(3, 0)$, $(0, 6)$
- Solution 1: $(2, 2)$: 20 patterns; 20 rolls used
- Solution 2: $(3, 0)$: 12, $(0, 6)$: 9, $(2, 2)$: 2 patterns: 23 rolls used

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- $W = 70, w_1 = 20, w_2 = 11, b_1 = 12, b_2 = 17$
- Feasible patterns: $\binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}, \binom{2}{1}, \binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{0}{3}, \binom{1}{3}, \binom{0}{4}, \binom{1}{4}, \binom{0}{5}, \binom{0}{6}$
- $x_1, \dots, x_{15} = \#$ of feasible patterns of the type $\binom{1}{0}, \dots, \binom{0}{6}$ respectively
-

$$\begin{aligned} \min \quad & x_1 + \cdots + x_{15} \\ \text{s.t.} \quad & x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cdots + x_{15} \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix} \\ & x_1, \dots, x_{15} \ge 0 \end{aligned}$$

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- Example: $2\begin{pmatrix} 0 \\ 6 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 5 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}$ 7 rolls used

  $4\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}$ 9 rolls used

- Any ideas?

3.3 Formulation

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Decision variables: x_j = number of rolls cut by pattern j characterized by vector $\boldsymbol{A_j}$:

$$\min \sum_{j=1}^{n} x_j$$
$$\sum_{j=1}^{n} \boldsymbol{A_j} \cdot x_j = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$
$$x_j \geq 0 \quad (\text{ integer})$$

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- Huge number of variables.
- Can we apply column generation, that is generate the patterns $\boldsymbol{A_j}$ on the fly?

3.4 Algorithm

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Idea: Generate feasible patterns as needed.

1) Start with initial patterns: $\begin{pmatrix} \lfloor \frac{W}{w_1} \rfloor \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lfloor \frac{W}{w_2} \rfloor \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{W}{w_3} \rfloor \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{W}{w_4} \rfloor \end{pmatrix}$

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2) Solve:

$$\min \; x_1 + \cdots + x_m$$
$$x_1 \boldsymbol{A_1} + \cdots + x_m \boldsymbol{A_m} = \boldsymbol{b}$$
$$x_i \geq 0$$

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3) Compute reduced costs

$\overline{c}_j = 1 - \boldsymbol{p}' \boldsymbol{A_j}$ for all patterns j

If $\overline{c}_j \geq 0$ current set of patterns optimal
If $\overline{c}_s < 0 \Rightarrow x_s$ needs to enter basis

How are we going to compute reduced costs $\overline{c}_j = 1 - \boldsymbol{p}' \boldsymbol{A_j}$ for all j? (huge number)

3.4.1 Key Idea

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4) Solve

$$\begin{aligned} z^* = \max \quad & \sum_{i=1}^{m} p_i a_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} w_i a_i \le W \\ & a_i \ge 0, \quad \text{integer} \end{aligned}$$

This is the integer knapsack problem

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- If $z^* \le 1 \Rightarrow 1 - \boldsymbol{p' A_j} > 0 \ \forall j \Rightarrow$ current solution optimal
- If $z^* > 1 \Rightarrow \exists\, s\!: \ 1 - \boldsymbol{p' A_s} < 0 \Rightarrow$ Variable x_s becomes basic, i.e., a new pattern $\boldsymbol{A_s}$ will enter the basis.
- Perform min-ratio test and update the basis.

3.5 Dynamic Programming

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$$\begin{aligned} F(u) = \max \quad & p_1 a_1 + \cdots + p_m a_m \\ \text{s.t.} \quad & w_1 a_1 + \cdots + w_m a_m \le u \\ & a_i \ge 0, \ \text{integer} \end{aligned}$$

- For $u \le w_{min}$, $F(u) = 0$.
- For $u \ge w_{min}$

$$F(u) = \max_{i=1,\ldots,m} \{p_i + F(u - w_i)\}$$

Why ?

3.6 Example

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$$\begin{aligned} \max \quad & 11x_1 + 7x_2 + 5x_3 + x_4 \\ \text{s.t.} \quad & 6x_1 + 4x_2 + 3x_3 + x_4 \le 25 \\ & x_i \ge 0, \quad x_i \text{ integer} \end{aligned}$$

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$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(2) &= 1 + F(1) = 2 \\ F(3) &= \max(5 + F(0)^*, 1 + F(2)) = 5 \\ F(4) &= \max(7 + F(0)^*, 5 + F(1), 1 + F(3)) = 7 \\ F(5) &= \max(7 + F(1)^*, 5 + F(2), 1 + F(4)) = 8 \\ F(6) &= \max(11 + F(0)^*, 7 + F(2), 5 + F(3), 1 + F(5)) = 11 \\ F(7) &= \max(11 + F(1)^*, 7 + F(2), 5 + F(3), 1 + F(4)) = 12 \\ F(8) &= \max(11 + F(2), 7 + F(4)^*, 5 + F(5), 1 + F(7)) = 14 \\ F(9) &= 11 + F(3) = 16 \\ F(10) &= 11 + F(4) = 18 \\ F(u) &= 11 + F(u - 6) = 16 \qquad u \ge 11 \end{aligned}$$

$\Rightarrow F(25) = 11 + F(19) = 11 + 11 + F(13) = 11 + 11 + 11 + F(7) = 33 + 12 = 45$

$\boxed{x^* = (4, 0, 0, 1)}$

4 Stochastic Programming

4.1 Example

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	Wrenches	Pliers	Cap.
Steel (lbs)	1.5	1.0	27,000
Molding machine (hrs)	1.0	1.0	21,000
Assembly machine (hrs)	0.3	0.5	9,000*
Demand limit (tools/day)	15,000	16,000	
Contribution to earnings ($/1000 units)	$130*	$100	

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$$\begin{array}{rl} \max & 130W + 100P \\ \text{s.t.} & W \leq 15 \\ & P \leq 16 \\ & 1.5W + P \leq 27 \\ & W + P \leq 21 \\ & 0.3W + 0.5P \leq 9 \\ & W, P \geq 0 \end{array}$$

4.1.1 Random data

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- Assembly capacity is random: $\begin{cases} 8000 & \text{with probability } \frac{1}{2} \\ 10,000 & \text{with probability } \frac{1}{2} \end{cases}$

- Contribution from wrenches: $\begin{cases} 160 & \text{with probability } \frac{1}{2} \\ 90 & \text{with probability } \frac{1}{2} \end{cases}$

4.1.2 Decisions

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- Need to decide steel capacity in the current quarter. Cost 58$/1000lbs.
- Soon after, uncertainty will be resolved.
- Next quarter, company will decide production quantities.

4.1.3 Formulation

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State	Cap.	W. contr.	Prob.
1	8,000	160	0.25
2	10,000	160	0.25
3	8,000	90	0.25
4	10,000	90	0.25

Decision Variables: S: steel capacity,
$P_i, W_i : i = 1, \ldots, 4$ production plan under state i. SLIDE 25

$$\begin{array}{rl}
\max & -58S + 0.25Z_1 + 0.25Z_2 + 0.25Z_3 + 0.25Z_4 \\
\text{s.t.} & \\
\text{Ass. 1} & 0.3W_1 + 0.5P_1 \leq 8 \\
\text{Mol. 1} & W_1 + P_1 \leq 21 \\
\text{Ste. 1} & -S + 1.5W_1 + P_1 \leq 0 \\
\text{W.d. 1} & W_1 \leq 15 \\
\text{P.d. 1} & P_1 \leq 16 \\
\text{Obj. 1} & -Z_1 + 160W_1 + 100P_1 = 0
\end{array}$$

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$$\begin{array}{rl}
\text{Ass. 2} & 0.3W_2 + 0.5P_2 \leq 10 \\
\text{Mol. 2} & W_2 + P_2 \leq 21 \\
\text{Ste. 2} & -S + 1.5W_2 + P_2 \leq 0 \\
\text{W.d. 2} & W_2 \leq 15 \\
\text{P.d. 2} & P_2 \leq 16 \\
\text{Obj. 2} & -Z_2 + 160W_2 + 100P_2 = 0
\end{array}$$

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$$\begin{array}{rl}
\text{Ass. 3} & 0.3W_3 + 0.5P_3 \leq 8 \\
\text{Mol. 3} & W_3 + P_3 \leq 21 \\
\text{Ste. 3} & -S + 1.5W_3 + P_3 \leq 0 \\
\text{W.d. 3} & W_3 \leq 15 \\
\text{P.d. 3} & P_3 \leq 16 \\
\text{Obj. 3} & -Z_3 + 90W_3 + 100P_3 = 0
\end{array}$$

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$$\begin{array}{rl}
\text{Ass. 4} & 0.3W_4 + 0.5P_4 \leq 10 \\
\text{Mol. 4} & W_4 + P_4 \leq 21 \\
\text{Ste. 4} & -S + 1.5W_4 + P_4 \leq 0 \\
\text{W.d. 4} & W_4 \leq 15 \\
\text{P.d. 4} & P_4 \leq 16 \\
\text{Obj. 4} & -Z_4 + 90W_4 + 100P_4 = 0 \\
 & S, W_i, P_i \geq 0
\end{array}$$

4.1.4 Solution

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Solution: $S = 27,250$lb.

	W_i	P_i
1	15,000	4,750
2	15,000	4,750
3	12,500	8,500
4	5,000	16,000

4.2 Two-stage problems

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- Random scenarios indexed by $w = 1, \ldots, k$. Scenario w has probability α_w.
- First stage decisions: $\boldsymbol{x}$: $\boldsymbol{Ax} = b, \boldsymbol{x} \geq \boldsymbol{0}$.
- Second stage decisions: y_w: $w = 1, \ldots, k$.
- Constraints:
  $\boldsymbol{B_w x} + \boldsymbol{D_w y_w} = \boldsymbol{d_w}$, $\boldsymbol{y_w} \geq \boldsymbol{0}$.

4.2.1 Formulation

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$$
\begin{array}{llllll}
\min & \boldsymbol{c}'\boldsymbol{x} + & \alpha_1 \boldsymbol{f}_1'\boldsymbol{y}_1 + & \cdots + & \alpha_k \boldsymbol{f}_k'\boldsymbol{y}_k & \\
 & \boldsymbol{Ax} & & & & = \boldsymbol{b} \\
 & \boldsymbol{B}_1\boldsymbol{x} + & \boldsymbol{D}_1\boldsymbol{y}_1 & & & = \boldsymbol{d}_1 \\
 & \boldsymbol{B}_2\boldsymbol{x} + & & \boldsymbol{D}_2\boldsymbol{y}_2 & & = \boldsymbol{d}_2 \\
 & \vdots & & \ddots & & \vdots \\
 & \boldsymbol{B}_k\boldsymbol{x} + & & & \boldsymbol{D}_k\boldsymbol{y}_k & = \boldsymbol{d}_k \\
 & \boldsymbol{x}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, \ldots, & \boldsymbol{y}_k & \geq \boldsymbol{0}.
\end{array}
$$

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Structure: $\boldsymbol{x}$ $\boldsymbol{y}_1$ $\boldsymbol{y}_2$ $\boldsymbol{y}_3$ $\boldsymbol{y}_4$