

# 15.093J Optimization Methods

## Lecture 5: Duality Theory I

# 1 Outline

SLIDE 1

- Motivation of duality
- General form of the dual
- Weak and strong duality
- Relations between primal and dual
- Economic Interpretation
- Complementary Slackness

# 2 Motivation

## 2.1 An idea from Lagrange

SLIDE 2

Consider the LOP, called the **primal** with optimal solution  $\mathbf{x}^*$

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Relax the constraint

$$\begin{aligned} g(\mathbf{p}) = \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{Ax}) \\ \text{s.t.} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$g(\mathbf{p}) \leq \mathbf{c}'\mathbf{x}^* + \mathbf{p}'(\mathbf{b} - \mathbf{Ax}^*) = \mathbf{c}'\mathbf{x}^*$$

Get the tightest lower bound, i.e.,

$$\max g(\mathbf{p})$$

$$\begin{aligned} g(\mathbf{p}) &= \min_{\mathbf{x} \geq \mathbf{0}} [\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{Ax})] \\ &= \mathbf{p}'\mathbf{b} + \min_{\mathbf{x} \geq \mathbf{0}} (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x} \end{aligned}$$

Note that

$$\min_{\mathbf{x} \geq \mathbf{0}} (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x} = \begin{cases} 0, & \text{if } \mathbf{c}' - \mathbf{p}'\mathbf{A} \geq \mathbf{0}', \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\mathbf{Dual} \quad \max g(\mathbf{p}) \quad \Leftrightarrow \quad \max_{\text{s.t. } \mathbf{p}'\mathbf{A} \leq \mathbf{c}'} \mathbf{p}'\mathbf{b}$$

### 3 General form of the dual

SLIDE 3

Primal	Dual
min $c'x$	max $p'b$
s.t. $a'_i x \geq b_i \quad i \in M_1$	s.t. $p_i \geq 0 \quad i \in M_1$
$a'_i x \leq b_i \quad i \in M_2$	$p_i \leq 0 \quad i \in M_1$
$a'_i x = b_i \quad i \in M_3$	$p_i \text{ free} \quad i \in M_3$
$x_j \geq 0 \quad j \in N_1$	$p'A_j \leq c_j \quad j \in N_1$
$x_j \leq 0 \quad j \in N_2$	$p'A_j \geq c_j \quad j \in N_2$
$x_j \text{ free} \quad j \in N_3$	$p'A_j = c_j \quad j \in N_3$

#### 3.1 Example

SLIDE 4

min $x_1 + 2x_2 + 3x_3$	max $5p_1 + 6p_2 + 4p_3$
s.t. $-x_1 + 3x_2 = 5$	s.t. $p_1$ free
$2x_1 - x_2 + 3x_3 \geq 6$	$p_2 \geq 0$
$x_3 \leq 4$	$p_3 \leq 0$
$x_1 \geq 0$	$-p_1 + 2p_2 \leq 1$
$x_2 \leq 0$	$3p_1 - p_2 \geq 2$
$x_3$ free,	$3p_2 + p_3 = 3.$

SLIDE 5

Primal	min	max	dual
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	$\geq 0$ $\leq 0$ $\text{free}$	variables
variables	$\geq 0$ $\leq 0$ $\text{free}$	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Theorem: The dual of the dual is the primal.

#### 3.2 A matrix view

SLIDE 6

min $c'x$	max $p'b$
s.t. $Ax = b$	s.t. $p'A \leq c'$
$x \geq 0$	
min $c'x$	max $p'b$
s.t. $Ax \geq b$	s.t. $p'A = c'$
	$p \geq 0$

### 4 Weak Duality

SLIDE 7

Theorem:

If  $x$  is primal feasible and  $p$  is dual feasible then  $p'b \leq c'x$

Proof

$$p'b = p'Ax \leq c'x$$

Corollary:

If  $\mathbf{x}$  is primal feasible,  $\mathbf{p}$  is dual feasible, and  $\mathbf{p}'\mathbf{b} = \mathbf{c}'\mathbf{x}$ , then  $\mathbf{x}$  is optimal in the primal and  $\mathbf{p}$  is optimal in the dual.

## 5 Strong Duality

SLIDE 8

Theorem: If the LOP has optimal solution, then so does the dual, and optimal costs are equal.

Proof:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Apply Simplex; optimal solution  $\mathbf{x}$ , basis  $\mathbf{B}$ .

Optimality conditions:

$$\mathbf{c}' - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A} \geq \mathbf{0}'$$

SLIDE 9

Define  $\mathbf{p}' = \mathbf{c}'_B \mathbf{B}^{-1} \Rightarrow \mathbf{p}'\mathbf{A} \leq \mathbf{c}'$

$\Rightarrow \mathbf{p}$  dual feasible for

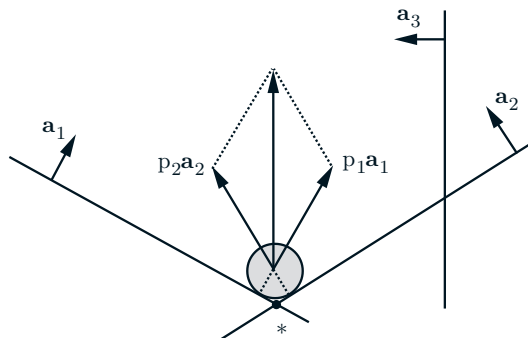
$$\begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} \\ \text{s.t.} \quad & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \end{aligned}$$

$$\mathbf{p}'\mathbf{b} = \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}'_B \mathbf{x}_B = \mathbf{c}'\mathbf{x}$$

$\Rightarrow \mathbf{x}, \mathbf{p}$  are primal and dual optimal

### 5.1 Intuition

SLIDE 10



## 6 Relations between primal and dual

SLIDE 11

	Finite opt.	Unbounded	Infeasible
Finite opt.	*		
Unbounded			*
Infeasible		*	*

## 7 Economic Interpretation

SLIDE 12

- $\mathbf{x}$  optimal nondegenerate solution:  $\mathbf{B}^{-1}\mathbf{b} > \mathbf{0}$
- Suppose  $\mathbf{b}$  changes to  $\mathbf{b} + \mathbf{d}$  for some small  $\mathbf{d}$
- How is the optimal cost affected?
- For small  $\mathbf{d}$  feasibility unaffected
- Optimality conditions unaffected
- New cost  $\mathbf{c}'_B \mathbf{B}^{-1}(\mathbf{b} + \mathbf{d}) = \mathbf{p}'(\mathbf{b} + \mathbf{d})$
- If resource  $i$  changes by  $d_i$ , cost changes by  $p_i d_i$ : “Marginal Price”

## 8 Complementary slackness

### 8.1 Theorem

SLIDE 13

Let  $\mathbf{x}$  primal feasible and  $\mathbf{p}$  dual feasible. Then  $\mathbf{x}, \mathbf{p}$  optimal if and only if

$$p_i(\mathbf{a}'_i \mathbf{x} - b_i) = 0, \quad \forall i$$

$$x_j(c_j - \mathbf{p}'\mathbf{A}_j) = 0, \quad \forall j$$

### 8.2 Proof

SLIDE 14

- $u_i = p_i(\mathbf{a}'_i \mathbf{x} - b_i)$  and  $v_j = (c_j - \mathbf{p}'\mathbf{A}_j)x_j$
- If  $\mathbf{x}, \mathbf{p}$  primal and dual feasible,  $u_i \geq 0, v_j \geq 0 \forall i, j$ .
- Also  $\mathbf{c}'\mathbf{x} - \mathbf{p}'\mathbf{b} = \sum_i u_i + \sum_j v_j$ .
- By the strong duality theorem, if  $\mathbf{x}$  and  $\mathbf{p}$  are optimal, then  $\mathbf{c}'\mathbf{x} = \mathbf{p}'\mathbf{b} \Rightarrow u_i = v_j = 0$  for all  $i, j$ .
- Conversely, if  $u_i = v_j = 0$  for all  $i, j$ , then  $\mathbf{c}'\mathbf{x} = \mathbf{p}'\mathbf{b}$ ,
- $\Rightarrow \mathbf{x}$  and  $\mathbf{p}$  are optimal.

### 8.3 Example

SLIDE 15

$$\begin{array}{ll} \min & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t.} & 5x_1 + x_2 + 3x_3 = 8 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{array} \qquad \begin{array}{ll} \max & 8p_1 + 3p_2 \\ \text{s.t.} & 5p_1 + 3p_2 \leq 13 \\ & p_1 + p_2 \leq 10 \\ & 3p_1 \leq 6 \end{array}$$

Is  $\mathbf{x}^* = (1, 0, 1)'$  optimal?

SLIDE 16

$$5p_1 + 3p_2 = 13, \quad 3p_1 = 6$$

$$\Rightarrow p_1 = 2, \quad p_2 = 1$$

Objective=19

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