8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

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Lecture 8

Reminder from last lecture

We consider correlation functions of gauge invariant (local) operators, including single-trace operators $\mathcal{O}_k$ and multiple-trace operators like $\mathcal{O}_m \mathcal{O}_n(x)$. Without loss of generality, we can focus on single-trace operators since multiple-trace ones are products of them.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \sum_{n=0}^{\infty} N^{2-n-2h} F_n^{(h)}(x_1,\cdots,x_n;\lambda) = O(N^{2-n}) + O(N^{-n}) + O(N^{-n-2}) + \cdots$$

The first term comes from planar diagrams, second one from torus diagrams, third one from double torus diagrams etc.

Physical implications:

1. In the large N limit, $\mathcal{O}(x)|0\rangle$ can be interpreted as creating a single-particle state ("glue ball"). Similarly $:\mathcal{O}_1\cdots\mathcal{O}_n(x):|0\rangle$ represents n-particle state.

2. Fluctuations of "glue balls" are suppressed, *i.e.* $\frac{\sqrt{\langle\mathcal{O}^2\rangle_c}}{\langle\mathcal{O}\rangle} \sim N^{-1}{\to}0$, if $\langle\mathcal{O}\rangle \neq 0$.

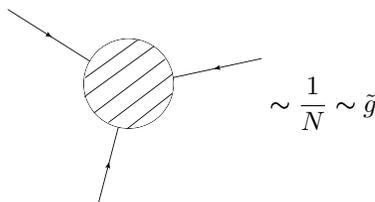In this lecture we first continue to give physical implications:

3. If we interpret

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \sim$$

$\sim O(N^{2-n}) + \cdots$
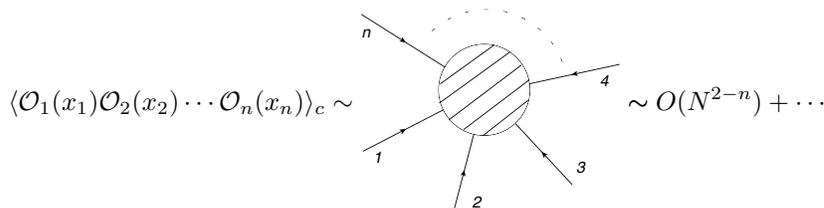
as "scattering amplitude" of n "glue balls", then to the leading order in $N{\to}\infty$, the scattering only involve tree-level interactions (only classical), among the glue ball states.

- Consider

$\sim \frac{1}{N} \sim \tilde{g}$

suppose we treat it as a basic vertex with coupling $\tilde{g}$, then the tree-level amplitude for n-particle scatterings scales as $\tilde{g}^{n-2} \sim N^{2-n}$.

- We can also include higher order vertices, but they should satisfy:

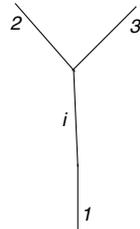$\sim \tilde{g}^2$, $\sim \tilde{g}^3$, $\cdots$

- There are no more than one-particle intermediate states. Consider *e.g.* $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(\frac{1}{N})$. If we insert a complete set of states at all possible places, due to large N counting, all states other than-single particle ones are suppressed:
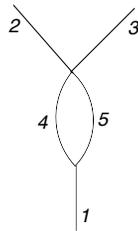
$$= \langle \mathcal{O}_1 : \mathcal{O}_4 \mathcal{O}_5 :\rangle\langle: \mathcal{O}_4 \mathcal{O}_5 : \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(N^{-3})$$

Compared to

$$= \langle \mathcal{O}_1 \mathcal{O}_i \rangle \langle \mathcal{O}_i \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(N^{-1})$$

i.e. all "loops" of glue balls are suppressed.

In summary, at leading order in $1/N$ expansion, we obtain a classical theory of glue balls, with interaction among glue balls given by $\tilde{g} \sim \frac{1}{N}$.

More explicitly

Gauge theory with finite $\hbar$ in the $N\to\infty$ limit	=	Glue ball theory with $\hbar\to 0$
Perturbative expansion in $\dfrac{1}{N}$	=	Loops of glue balls perturbative in $\hbar$

We will now show that these resemble a string theory.

1.5: LARGE N EXPANSION AS A STRING THEORY

QFT can be considered as a theory of "particles". The standard quantization approach is second quantization. In the first quantization approach, we directly quantize the motion of a particle in spacetime.
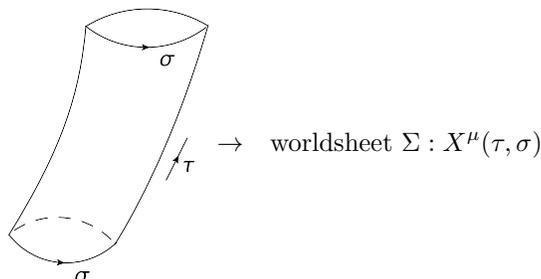

$$\to X^\mu(\tau)$$
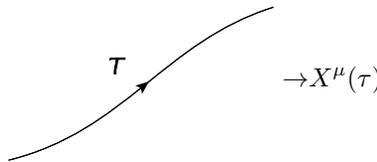
We have

$$Z = \int DX^\mu(\tau) e^{iS_{particle}}$$

where

$$S_{particle} = m \int dl = m \int d\tau \frac{dl}{d\tau} = m \int d\tau \sqrt{g_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{\tau}}$$

If we want to include interactions like $\lambda\phi^3$, we need to add them by hand.

In string theory, similarly, we need to quantize the motions of strings in spacetime.

Take a similar quantization approach

$$Z = \int DX^\mu(\tau) e^{iS_{string}} \tag{1}$$

The simplest form of S_{string} is the Nambu-Goto action

$$S_{NG} = T \int_\Sigma dA$$

here $T = \frac{1}{2\pi\alpha'}$ is the string tension (mass per unit length). $dA = \sqrt{-\det h_{ab}}d\sigma d\tau$ is the infinitesimal area of the world sheet with the induced matrix $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

To "define" and evaluate Eq. (1), the most convenient way is to go to Euclidean signature. For vacuum processes:

$$Z_{string} = \sum_{\text{all closed surfaces}} e^{-S_{NG}} = \sum_{h=0}^{\infty} e^{-\lambda\chi} \sum_{\text{surface with given topology}} e^{-S_{NG}}$$

here $\chi = 2 - 2h$ denotes the weight for different topologies, λ can be thought as the "chemical potential" for topology. If we define $g_s = e^\lambda$, the vacuum includes diagrams like
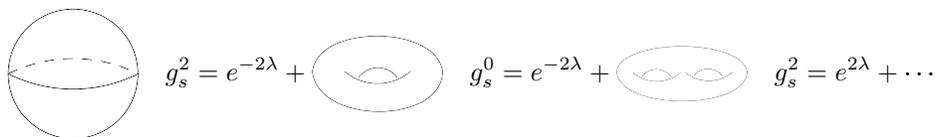
There is a remarkable fact about string theory: summing over topology of all surfaces automatically includes interactions of strings. In fact this fully specifies string interactions with no freedom of making arbitrary choices. To see this
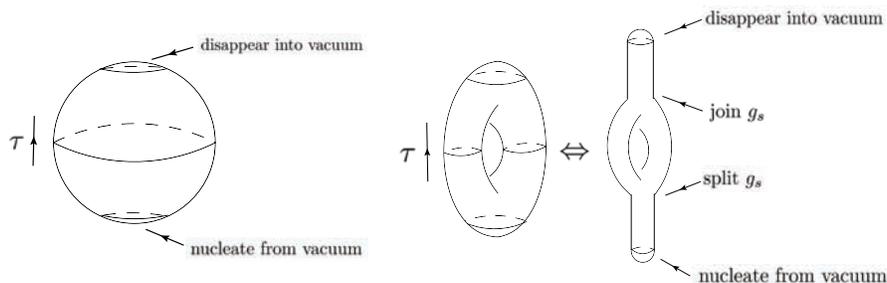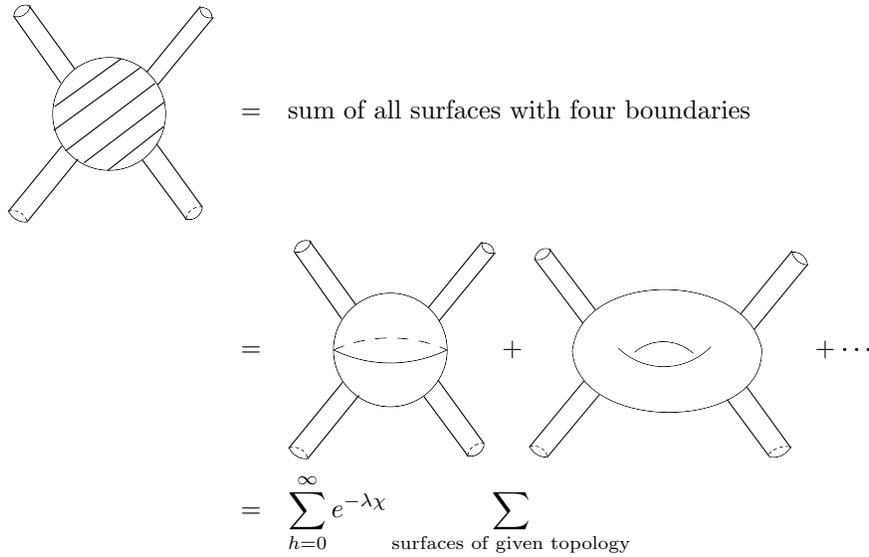

The surface can be thought as the vacuum bubble, at the south pole the string nucleates from vacuum and at the north pole, the string disappears into the vacuum. The torus can be thought as the one loop diagram, the string split into two strings and then join together again with interaction strength g_s on each vertex. Thus the basic interaction vertices are the splitting and rejoining of the strings, the coupling strength is $g_s = e^\lambda$:
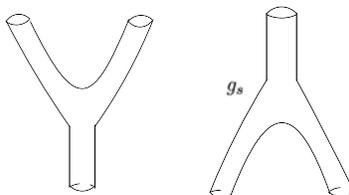
Now we include external strings, *e.g.*

$$\text{string} + \text{string} \quad \rightarrow \quad \text{string} + \text{string}$$

In the diagrammatic language:

$$ = \quad \text{sum of all surfaces with four boundaries} $$

$$ = \sum_{h=0}^{\infty} e^{-\lambda\chi} \sum_{\text{surfaces of given topology}} $$

where $\chi = 2 - 2h - n$, where n is the number of boundaries (number of external strings).

Thus for n-string scattering process (including vacuum processes, *i.e.* n=0)

$$ A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} F_n^{(h)} = g_s^{n-2} F_n^{(0)} + g_s^n F_n^{(1)} + g_s^{n+2} F_n^{(2)} + \cdots $$

The first term comes from tree-level diagrams (sphere topology), second term comes from 1-loop diagrams (torus topology), third term comes from 2-loop ones (double-torus topology) etc.

Now comparing with the large N expansion of a gauge theory as we discussed earlier (including n=0)

$$ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_c = \sum_{h=0}^{\infty} N^{2-n-2h} f_n^{(h)} = N^{2-n} f_n^{(0)} + N^{-n} f_n^{(1)} + N^{-n-2} f_n^{(2)} $$

The first term comes from planar diagrams (sphere topology), second one comes from torus diagrams, third one comes from double-torus diagrams, etc.

We see an identical mathematical structure of the two theories with the identification:

$$ e^\lambda = g_s \quad \leftrightarrow \quad \frac{1}{N} $$
$$ \text{external strings} \quad \leftrightarrow \quad \text{"glue balls" (single-trace operator) } \mathcal{O}_i(x)|0\rangle $$
$$ \text{sum over string world sheet of given topology} \quad \leftrightarrow \quad \text{sum over Feynman diagrams of given topology} $$
$$ \text{topology of the worldsheet} \quad \leftrightarrow \quad \text{topology of Feynman diagrams} $$

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