Chapter 3: Duality Toolbox

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Lecture 20

3.1.6: EUCLIDEAN CORRELATION FUNCTIONS

Let us consider Euclidean space. The basic observables of a CFT is correlation functions of local operators. From path integral formalism, we know a way to calculate them is by generating functional:

$$Z_{CFT}[\phi(x)] = \langle e^{\int d^d x \phi(x)\mathcal{O}(x)} \rangle_E \tag{1}$$

where subscript E denotes the path integral defined in Euclidean space and $\phi(x)$ and $\mathcal{O}(x)$ should be considered as the collection of all boundary operators and their sources.

The simplest case is $\phi(x) = 0$, then $Z_{CFT}[0]$ reduces to the partition function of CFT, which, by AdS/CFT correspondence, should be equal to the gravity dual partition function $Z_{gravity}$. Since we have discussed the correspondence:

$$\mathcal{O}(x) \iff \Phi_n(x,z) \tag{2}$$

$$\phi(x) \iff \Phi_{nn}(x,z)|_{\partial AdS} \tag{3}$$

where n is short for normalizable and nn is for non-normalizable, we should expect

$$Z_{CFT}[\phi(x)] = Z_{bulk}[\Phi|_{\partial AdS} = \phi(x)] \tag{4}$$

where $\Phi|_{\partial AdS} = \phi(x)$ really means $\Phi(z,x) \to \phi(x) z^{d-\Delta}$ when $z \to 0$. Generally we do not know how to define $Z_{bulk}[\Phi|_{\partial AdS}]$, but one can define it in semi-classical limit: $g_s \to 0$ and $\alpha' \to 0$. The path integral formalism gives

$$Z_{bulk}[\Phi|_{\partial AdS}] = \int_{\Phi|_{\partial AdS} = \phi(x)} D\Phi e^{S_E[\Phi]} \tag{5}$$

where the action is

$$S_E[\Phi] = -\frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{g}(\mathcal{R} + \mathcal{L}_{matter}) \tag{6}$$

In the semi-classical limit, the leading order partition function is given by its stationary point, namely the classical solution:

$$Z_{bulk} = \exp(S_E[\Phi_c]) \tag{7}$$

where Φ_c is the classical solution to equation of motion that satisfies the boundary conditions. As we know before, this limit corresponds to $N \to \infty$ and $\lambda \to \infty$ limit in the boundary theory. In particular, $S_E[\Phi_c] \propto \frac{1}{2\kappa^2} \propto N^2$ is consistent with the leading behavior of a large N gauge theory.

Thus in $N \to \infty$, $\lambda \to \infty$ limit, we have the following crucial formula

$$\log Z_{CFT}[\phi] = S_E[\Phi_c] \Phi_c \to z^a \phi(x) \quad (z \to 0) \tag{8}$$

where a depends on dimension and spin of an operator, e.g. $a = d - \Delta$ for scalar operator of dimension Δ; $a = 0$ for conserved current; $a = 2$ for stress tensor.

Remarks

1. In the generating functional, ϕ in $\int d^d x \phi(x)\mathcal{O}(x)$ should be considered as infinitesimal correction, i.e. $\log Z_{CFT}[\phi]$ should be considered as a power series of $\phi(x)$. Similar for the right hand side of (8), where the non-normalizable mode of Φ should be infinitesimal and created perturbatively.

2. Both sides of (8) are in fact divergent. The left hand side has usual UV divergences of a QFT, which by IR/UV connection, corresponds to the volume divergences of AdS on the right hand side. Thus in order to obtain finite answer, we need to renormalize them. The approach is to set a cutoff of $z = \epsilon$ when $z \to 0$ and add a local functional contact term. To be specific, we have

$$S_E^{(R)}[\Phi_c] = S_E[\Phi_c]|_{z=\epsilon} + S_{ct}[\Phi_c(\epsilon)] \tag{9}$$

3. With renormalized action, general connected n-point function is given by functional derivative with respect to ϕ:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_c = \left.\frac{\delta^n \log Z^{(R)}_{CFT}}{\delta\phi_1(x_1)\cdots\delta\phi_n(x_n)}\right|_{\phi=0} = \left.\frac{\delta^n S^{(R)}_E[\Phi_c]}{\delta\phi_1(x_1)\cdots\delta\phi_n(x_n)}\right|_{\phi=0} \tag{10}$$

4. Especially 1-point function is nonzero in the presence of source ϕ.

$$\langle\mathcal{O}(x)\rangle_\phi = \frac{\delta S^{(R)}_E[\Phi_c]}{\delta\phi(x)} = \lim_{z\to 0} z^{d-\Delta}\frac{\delta S^{(R)}_E[\Phi_c]}{\delta\Phi_c(z,x)} \tag{11}$$

Note $\frac{\delta S^{(R)}_E[\Phi_c]}{\delta\Phi_c(z,x)} \sim \Pi_c(z,x)$ is the canonical momentum conjugate to Φ treating z as "time" (later we will show this for scalar field). Now we get

$$\langle\mathcal{O}(x)\rangle_\phi \sim \lim_{z\to 0} z^{d-\Delta}\Pi^{(R)}_c(z,x) \tag{12}$$

One can show that for $\Phi(z,x)\to z^{d-\Delta}A(x)+z^\Delta B(x)$ when $z\to 0$,

$$\langle\mathcal{O}(x)\rangle_\phi = 2\nu B(x) \qquad (\nu = \Delta - \frac{d}{2}) \tag{13}$$

namely the 1-point function corresponds to the normalizable mode function. One can check that $\langle\mathcal{O}(x)\rangle_\phi$ and $B(x)$ do have the same transformation property under scaling: $\langle\mathcal{O}(\lambda x)\rangle_\phi = \lambda^{-\Delta}\langle\mathcal{O}(x)\rangle_\phi$ and $\Phi(\lambda z,\lambda x) = \Phi(z,x) \overset{z\sim 0}{\Longrightarrow} \lambda^\Delta z^\Delta B(\lambda x) = z^\Delta B(x) \implies B(\lambda x) = \lambda^{-\Delta}B(x)$.

Let us calculate a massive scalar field as an example to show the above equality is valid. The Euclidean action is given by

$$S_E = -\frac{1}{2}\int d^dxdz\sqrt{g}(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi^2) \tag{14}$$

and metric is

$$ds^2 = \frac{R^2}{z^2}(dz^2+dx^2) \qquad dx^2 \equiv \eta_{\mu\nu}dx^\mu dx^\nu \tag{15}$$

Our goal is to find $S^{(R)}_E[\Phi_c]$ for $\lim_{z\to 0} z^{\Delta-d}\Phi_c(x) = \phi(x)$. Take the functional derivative with respect to $\partial_z\Phi$ to Lagrangian, we get the canonical momentum as $\Pi = -\sqrt{g}g^{zz}\partial_z\Phi$. The equation of motion is

$$-\partial_M(\sqrt{g}g^{MN}\partial_N\Phi) + m^2\Phi = 0 \tag{16}$$

We write $\Phi = \Phi(k,z)e^{ikx}$ and plug into the equation of motion to get

$$z^{d+1}\partial_z(z^{1-d}\partial_z\Phi) - k^2z^2\Phi - m^2R^2\Phi = 0 \tag{17}$$

This equation can be exactly solved, but we will not need it until the last step. Since we are calculating the effective action with classical solution Φ, we get

$$\begin{aligned} S_E[\Phi_c] &= -\frac{1}{2}\int d^{d+1}x\Phi_c(-\partial_M(\sqrt{g}g^{MN}\partial_N\Phi)+m^2\Phi) - \frac{1}{2}\int_0^\infty dzd^dx\partial_M(\sqrt{g}g^{MN}\partial_N\Phi_c\Phi_c) \\ &= \frac{1}{2}\int d^dx\Pi_c(z,x)\Phi_c(z,x)\Big|_0^\infty \\ &= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\Phi_c(k,z)\Pi_c(-k,z)\Big|_0^\infty \end{aligned} \tag{18}$$

where in the second line we used the equation of motion and it shows that $\frac{\delta S^{(R)}_E[\Phi_c]}{\delta\Phi_c(z,x)} \sim \Pi_c(z,x)$ in the limit $z\to 0$.

First look at its behavior as $z\to\infty$. Since we should require the regularity of Φ in the bulk, Φ must be finite when $z\to\infty$, so is $\partial_z\Phi$. However, we know $\Pi = -\sqrt{g}g^{zz}\partial_z\Phi \sim z^{1-d}\partial_z\Phi \to 0$ when $z\to\infty$. Then the term in infinity vanishes. For $z\to 0$, the behavior is

$$\Phi_c \to A(x)z^{d-\Delta} + B(x)z^\Delta \tag{19}$$

and

$$\Pi = -\sqrt{g} g^{zz} \partial_z \Phi \to -(d-\Delta) A(x) z^{-\Delta} - \Delta B(x) z^{\Delta - d} \tag{20}$$

If we restrict ourselves with $\Delta = \frac{d}{2} + \nu \geq \frac{d}{2}$, the we see $\Phi_c \Pi_c$ is divergent because of $z^{d-2\Delta}$ term (for $\Delta = \frac{d}{2}$ requires special treatment), so is $S_E[\Phi_c]$. Then we need to introduce a contact term to cancel it. As we discussed before, the renormalized action is

$$S_E[\Phi_c] = -\frac{1}{2} \int d^d x \Pi_c \Phi_c \bigg|_\epsilon + S_{ct}[\Phi_c(\epsilon, x)] \tag{21}$$

where the contact term can be chosen as

$$S_{ct}[\Phi_c(\epsilon, x)] = \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} f(k^2) \Phi_c(k, z) \Phi_c(-k, z) \tag{22}$$

where $f(k^2)$ is some analytic function in k^2 such that to cancel the divergence and then taking the limit $\epsilon \to 0$. For this purpose, we will need to consider subleading terms in (19). Consider the basis of solutions Φ_1 and Φ_2 defined by

$$\Phi_1 \to z^{d-\Delta} \qquad \Phi_2 \to z^{\Delta} \qquad (z \to 0) \tag{23}$$

For small z,

$$\Phi_1 = z^{d-\Delta}(1 + a_1 k^2 z^2 + \cdots) \tag{24}$$

$$\Phi_2 = z^{\Delta}(1 + b_1 k^2 z^2 + \cdots) \tag{25}$$

where each expansion is analytic in k^2 because the equation of motion depends only on k^2. The canonical momenta basis are given by

$$\Pi_1 \to -(d-\Delta) z^{-\Delta} \tag{26}$$

$$\Pi_2 \to -\Delta z^{\Delta - d} \tag{27}$$

Our general solution should be expanded upon those basis as

$$\Phi_c = A(k)\Phi_1 + B(k)\Phi_2 \tag{28}$$

$$\Pi_c = A(k)\Pi_1 + B(k)\Pi_2 \tag{29}$$

Substitute this to the action, we get

$$S_E[\Phi_c]|_{z=\epsilon} = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(A^2 \Pi_1 \Phi_1 + B^2 \Pi_2 \Phi_2 + AB(\Pi_1 \Phi_2 + \Phi_1 \Pi_2) \right) \tag{30}$$

where the notation is $A^2 \equiv A(k)A(-k)$ as so on. Note in above formula, the first term is divergent, the second is vanishing and the last one is finite. This implies that we can choose our contact term as

$$\begin{aligned} S_{ct} &= \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} \frac{\Pi_1}{\Phi_1} \Phi^2 \\ &= \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} \frac{\Pi_1}{\Phi_1} (A\Phi_1 + B\Phi_2)^2 \\ &= \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} \left(A^2 \Phi_1 \Pi_1 + 2AB \Pi_1 \Phi_2 + B^2 \frac{\Pi_1}{\Phi_1} \Phi_2^2 \right) \end{aligned} \tag{31}$$

where $\Pi_1/\Phi_1|_{z=\epsilon}$ is an analytic function in k^2. We see the first term is the divergent to cancel the one in S_E, the second term is finite and the last term is vanishing. Sum them together and we get the renormalized action:

$$S_E^{(R)}[\Phi_c] = \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} A(-k) B(k) (\Pi_1 \Phi_2 - \Phi_1 \Pi_2) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} 2\nu A(-k) B(k) \tag{32}$$

Imposing the boundary condition $A(k) = \phi(k)$ and the regularity of Φ at $z \to \infty$ will fix the ration $\chi(k) = B(k)/A(k)$ (for this part we have to solve the equation of motion explicitly), we see

$$S_E^{(R)}[\Phi_c] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} 2\nu \chi(k) \phi(k) \phi(-k) \tag{33}$$

where $\chi(k)$ is independent on $\phi(k)$ and which implies

$$\langle \mathcal{O}(k) \rangle_\phi = 2\nu \chi(k) \phi(k) = 2\nu B(k) \tag{34}$$

or in coordinate space:

$$\langle \mathcal{O}(x) \rangle_\phi = 2\nu B(x) \tag{35}$$

From this formula, we can also calculate 2-point function

$$G_E(k) = \left. \frac{\delta^2 S_E^{(R)}}{\delta\phi(k)\delta\phi(-k)} \right|_{\phi=0} = 2\nu\chi(k) = \langle \mathcal{O}(x) \rangle_\phi / \phi(k) \tag{36}$$

In other words, this is a linear response of 1-point function to the source $\phi(k)$,

$$\langle \mathcal{O}(x) \rangle_\phi = G_E(k) \phi(k) \tag{37}$$

If we solve out the equation of motion explicitly, we can find χ such that we can calculate the two point function from bulk point of view. The solution is Bessel function if we require regularity at $z \to \infty$,

$$\Phi_c \propto z^{d/2} K_\nu(kz) \tag{38}$$

as $z \to 0$,

$$K_\nu(kz) = \frac{\Gamma(\nu)}{2} \left(\frac{kz}{2} \right)^{-\nu} (1 + \cdots) + \frac{\Gamma(-\nu)}{2} \left(\frac{kz}{2} \right)^{\nu} (1 + \cdots) \tag{39}$$

which implies

$$\chi(k) = \frac{B}{A} = \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2} \right)^{2\nu} \implies G_E(k) = 2\nu \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2} \right)^{2\nu} \tag{40}$$

We can do the Fourier transformation back to coordinate space and find

$$G_E(x) \propto \frac{1}{|x|^{2\Delta}} \tag{41}$$

which is consistent with the 2-point function of CFT of a scalar with scaling dimension Δ.

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