PROFESSOR: So last time, we started looking at the system of [? spins. ?] So there was a field S of x on the lattice. And the energy cost was proportional to differences of space on two neighboring sites, which if we go to the continuum, became something like gradient of the vector S. You have to integrate this, of course, over all space.

We gave this a rate of K over 2. There was some energy costs [INAUDIBLE] of this [? form ?] so that the particular configuration was weighted by this factor. And to calculate the partition function, we had to integrate over all configurations of this over this field S.

And the constraint that we had was that this was a unit vector so that this was an n component field whose magnitude was 1, OK? So this is what we want to calculate. Again, whenever we are writing an expression such as this, thinking that we started with some average system [INAUDIBLE] some kind of a coarse graining. There is a short distance [INAUDIBLE] [? replacing ?] all of these tiers.

Now what we can do is imagine that this vector S in its ground state, let's say, is pointing in some particular direction throughout the system and that fluctuations around this ground state in the transverse direction are characterized by some vector of pi that is n minus 1 dimensional. And so this partition function can be written entirely [? rather ?] than fluctuations of the unit vector S in terms of the fluctuations of these transverse [? coordinates ?] [INAUDIBLE] pi. And we saw that the appropriate weight for this n minus 1 component vector of pi has within it a factor of something like square root of 1 minus pi squared. There's an overall factor of 2, but it doesn't matter.

And essentially, this says that because you have a unit vector, this pi cannot get too

big. You have to pay [? a cost ?] here, certainly not larger than 1. And then the expression for the energy costs can be written in terms of two parts [? where it ?] is the gradient of this vector pi.

But then there's also the gradient in the other direction which has magnitude square root of 1 minus pi squared. So we have this gradient squared. And so basically, these are two ways of writing the same thing, OK?

So we looked at this and we said that once we include all of these terms, what we have here is a non-linear [? activity ?] that includes, for example, interactions among the various modes. And one particular leading order term is if we expand this square root of 1 minus pi squared, if it would be something like pi square root of pi, so a particular term in this expansion as the from pi [? grad ?] pi multiplied with pi [? grad ?] pi. [INAUDIBLE], OK?

So a particular way of dealing these kinds of theories is to regard all of these things as interactions and perturbations with respect to a Gaussian weight which we can compute easily. And then you can either do that perturbation straightforwardly or from the beginning to a perturbative [? origin, ?] which is the route that we chose.

And this amount to changing the short distance cut off that we have here that is a to be b times a and averaging over all [? nodes ?] within that distance short wavelength between a and ba. And once we do that, we arrive at a new interaction. So the first step is to do a coarse graining between the range a and ba.

But then steps two and three amount to a rescaling in position space so that the cut-off comes back to ba and the corresponding thing in the spin space so that we start with a partition function that describes unit vectors. And after this transformation, we end up with a new partition function that also describes unit vectors so that after all of these three procedures, we hope that we are back to exactly the form that we had at the beginning with the same cut-off, with the same unit vector constraint, but potentially with a new interaction parameter K

And calculating what this new K is after we've scaled by a factor of b, the parts that

correspond to 2 and 3 are immediately obvious. Because whenever I see x, I have to replace it wit bx prime. And so from integration, I then get a factor of b to the d from the two gradients, I would get a factor of minus 2.

So the step that corresponds to this is trivial. The step that corresponds to replacing S with zeta S prime is also trivial. And it will give you zeta squared. Or do I have yet to tell you what is? We'll do that shortly.

And finally, the first step, which was the coarse graining, we found that what it did was that it replaced K by a strongly-- sorry. I didn't expect this. All right-- the factor K, which is larger by a certain amount. And the mathematical justification that I gave for this is we look at this expression, and we see that in this expression, each one of these pi's can be a long wavelength fluctuation or a short wavelength fluctuation.

Among the many possibilities is when these 2 pi's that are sitting out front correspond to the short wavelength fluctuations. These correspond to the long wavelength fluctuations. And you can see that averaging over these two will generate an interaction that looks like gradient of pi lesser squared. And that will change the coefficient over here by an amount that is clearly proportional to the average of pi greater squared. And that we can see in Fourier space is simply 1 over Kq squared over KK squared for modes that have wave number K.

So if I, rather than write this in real space, I write it in Fourier space, this is what I would get for the average. And in real space, I have to integrate over this K appropriately [INAUDIBLE] within the wave numbers lambda over [? real ?] [? light. ?] And this is clearly something that is inversely proportional to K. And the result of this integration of 1 over K squared-- we simply gave a [? 9 ?], which was i sub d of b. Because it depends on the dimension. It depends on [? et ?] [? cetera. ?]

AUDIENCE: Sir?

PROFESSOR: Yes?

AUDIENCE: Shouldn't there be an exponential inside the integral?

PROFESSOR: Why should there be an exponential inside the integral?

AUDIENCE: Oh, I thought we were Fourier transforming.

PROFESSOR: OK, it is true, when we Fourier transform for each pi, we will have a factor of e to the IK. If we have 2 pi's, I will get e to the IK e to the IK prime. But the averaging [INAUDIBLE] set K and K prime can be opposite each other. So the exponentials disappear. So always remember the integral of any field squared in real space is the same thing as the integral of that field squared in Fourier space. This is one of the first theorems of Fourier transformation.

OK, so this is a correction that goes like [INAUDIBLE]. And last time, to give you a kind of visual demonstration of what this factor is, I said that it is similar, but by no means identical to something like this, which is that a mode by itself has very low energy. But because we have coupling among different modes, here for the Goldstone modes of the surface, but here for the Goldstone modes of the spin, the presence of a certain amount of short range fluctuations will stiffen the modes that you have for longer wavelengths.

Now, I'm not saying that these two problems are mathematically identical. All I'm showing you is that the coupling between the short and long wavelength modes can lead to a stiffening of the modes over long distances because they have to fight off the [? rails ?] that have been established by shorter wavelength modes. You have to try to undo them. And that's an additional cost, OK?

Now, that stiffening over here is opposed by a factor of zeta over here. Essentially, we said that we have to ensure that what we are seeing after the three steps of RG is a description of a [? TOD ?] that has the same short distance cut-off and the same length so that the two partition functions can map on to each other. And again, another visual demonstration is that you can decompose the spin over here to a superposition of short and long wavelength modes. And we are averaging over these short wavelength modes.

And because of that, we will see that the effective length once that averaging has

been performed has been reduced. It has been reduced because I will write this as 1 minus pi squared over 2 to the lowest order. And pi squared has n minus 1 components. So this is n minus 1 over 2. And then I have to integrate over all of the modes pi alpha of K in this range. So I'm performing exactly the same integral as above. So the reduction is precisely the same integral as above, OK?

So the three steps of RG performed for this model to lowest order in this inverse K, our [? temperature-like ?] variable is given by this one [INAUDIBLE] once I substitute the value of zeta over there. So you can see that the answer K prime of b is going to b to the b minus 2 zeta squared-- ah, that's right. For zeta squared, essentially the square of that, 1 minus n minus 1 over K, the 2 disappears once I squared it.

Id of b divided by K-- that comes from zeta squared. And from here I can get the plus Id of be over K. And the whole thing gets multiplied by this K. And there is still terms at the order of temperature of 1 over K squared, OK?

And finally, we are going to do the same choice that we were doing for our epsilon expansion. That is, choose a rescaling factor that is just slightly larger than 1. Yes?

AUDIENCE: Sir, you're at n minus 1 over K times Id over K.

PROFESSOR: Yeah, I gave this too much. Thank you. OK, thank You. And we will write K prime at Kb to be K plus delta l dK by dl. And we note that for calculating this Id of b, when b goes to 1 plus delta l, all I need to do is to evaluate this integrand essentially on the shell. So what I will get is lambda to the power of d minus 2. The surface area of a unit sphere divided by 2 pi to the d, which is the combination that we have been calling K sub d, OK?

So once I do that, I will get that the dK by dl, OK? What do I get? I will get a d minus 2 here times K. And then I will get these two factors. There's n minus 1. And then there's 1 here. So that becomes n minus 2. I have a Idb, which is Kd lambda to the d minus 2. And then the 1 over K and K disappear. And so that's the expression that we have. Yes?

AUDIENCE: Sorry, is the Kd [? some ?] angle factor again?

PROFESSOR: OK, so you have to do this integration, which is written as the surface area inside an angle, K to the d minus 1 dK divided by 2 pi to the d. This is the combination that we have always called K sub d.

AUDIENCE: OK.

PROFESSOR: OK? Now, it actually makes more sense since we are making a low temperature expansion to define a T that is simply 1 over K. Its again, dimensionless. And then clearly dT by dl is going to be minus K squared dK by dl minus 1 over K squared.

Minus 1 over K squared becomes minus T squared dK by dl. So I just have to multiply the expression that I have up here with minus T squared, recognizing that TK is 1. So I end up with the [? recursion ?] convention for T, which is minus d minus 2T. And then it becomes plus n minus 2 Kd lambda to the d minus 2 T squared. And presumably, there are high order terms that we have not bothered to calculate. So this is the [INAUDIBLE] we focused on. OK?

So let's see whether this expression makes sense. So if I'm looking at dimensions that are less than 2, then the linear term in the expression is positive, which means that if I'm looking at the temperature axis and this is 0, and I start with a value that is slightly positive, because of this term, it will be pushed larger and larger values. So you think that you have a system at very low temperatures.

You look at it at larger and larger scales, and you find that it becomes effectively something that has higher temperature and becomes more and more disordered. So basically, this is a manifestation of something that we had said before, Mermin-Wagner theorem, which is no [? long ?] range order in d less than 2, OK?

Now, if I go to the other limit, d greater than 2, then something interesting happens, in that the linear term is negative. So if I start with a sufficiently small temperature or a large enough coupling, it will get stronger as we go towards an ordered phase, whereas the quadratic term for n greater than 2 has the opposite sign-- this is n greater than 2-- and pushes me towards disorder, which means that there should

be a fixed point that separates the two behaviors.

Any temperature lower than this will give me an ordered phase. Any temperature higher than this will give me a disordered phase. And suddenly, we see that we have potentially a way of figuring out what the phase transition is because this T star is a location that we can perturbatively access. Because we set this to 0, and we find that T star is equal to d minus 2 divided by n minus 2 Kd lambda to the d minus 2.

So now in order to have a theory that makes sense in the sense of the perturbation that we have carried out, we have to make sure that this is small. So we can do that by assuming that this quantity d minus 2 is a small quantity in making an expansion in d minus 2, OK? So in particular, T star itself we expect to be related to transition temperature, not something that is universal. But exponents are universal.

So what we do is we look at d by dl of delta T. Delta T is, let's say, T minus T star in one direction or the other. [INAUDIBLE]. And for that, what I need to do it to linearize this expression. So I will get a minus epsilon from her. And from here, I will get 2 n minus 2 Kd lambda to the d minus 2 T star times delta T.

I just took the derivative, evaluated the T star. And we can see that this combination is precisely the combination that I have to solve for T star. So this really becomes another factor of epsilon. I have minus epsilon plus [? 2 ?] epsilon. So this is epsilon delta T. So that tells me that my thermal eigenvalue is epsilon, a disorder clearly independent of n.

Now, we've seen that in order to fully characterize the exponent, including things like magnetization, et cetera, it makes sense to also put a magnetic field direction and figure out how rapidly you go along the magnetic field direction. So for that, one way of doing this is to go and add to this term, which is h integral S of x. And you can see very easily that under these steps of the transformation, essentially the only thing that happens is that I will get h prime at scale d is h from the integration. I will get a factor of b to the d. From the replacement of s with s prime, I will get a factor of zeta.

So this combination is simply my yh. And just bringing a little bit of manipulation will tell you that yh is d minus the part that comes from zeta, which is n minus 1 over 2 ld of b, which is lambda to the d minus 2 Kd. And then we have T star.

And again, you substitute for lambda to the d minus 2 Kd T star on what we have over here. And you get this to be d minus n minus 1 over 2n minus 2 epsilon. And again, to be consistent to order of epsilon, this d you will have to replace with 2 plus epsilon. And a little bit of manipulation will give you yh, which is 2 minus n minus 3 divided by 2n minus 2 epsilon.

I did this calculation of the two exponents rather rapidly. The reason for that is they are not particularly useful. That is, whereas we saw that coming from four dimensions the epsilon expansion was very useful to give us corrections to the [? mean ?] field values of 1/2, for example, for mu to order of 10% or so already by setting epsilon equals to 1. If I, for example, put here epsilon equal to 1 to [? access ?] 3 dimensions, I will conclude that mu, which is the inverse of yT is 1 in 3 dimensions independent of n. And let's say for super fluid it is closer to 2/3.

And so essentially, this expansion in some sense is much further away from 3 dimensions than the 4 minus epsilon coming from 4 dimensions, although numerically, we would have said epsilon equaled 1 to both of them. So nobody really has taken much advantage of this 2 plus epsilon expansion. So why is it useful?

The reason that this is useful is the third case that I have not explained, yet, which is, what happens if you sit exactly in 2 dimensions, OK? So if we sit exactly in 2 dimensions, this first term disappears. And you can see that the behavioral is determined by the second order and [? then ?] depends on the value of n.

So if I look at n, let's say, that is less than 2, then what I will see is that along the temperature axis, the quadratic term-- the linear term is absent-- the quadratic term, let's say, for n equals 1 is negative. And you're being pushed quadratically very slowly towards 0. The one example that we know is indeed n equals 1, the Ising

model. And we know that the Ising model in 2 dimensions has an ordered phase. It shouldn't really even be described by this because there are no Goldstone modes.

But n greater than 2, like n equals 3-- the Heisenberg model, is interesting. And what we see is that here the second order term is positive. And it is pushing you towards high temperatures. So you can see a disordered behavior.

And what this calculation tells you that is useful that you wouldn't have known otherwise is, what is the correlation [? length? ?] Because the recursion relation is now dT by dl is, let's say, n minus 2. And my d equals to 2. The lambda to the d minus 2, I can ignore. Kd is 2 pi from the [? solid ?] angle divided by 2 pi squared. So that's 1 over 2 pi times T squared, OK? So that's the recursion relation that we are dealing with.

I can divide by 1 over T squared. And then this becomes d by dl of minus 1 over T equals n minus 2 divided by 2 pi. I can integrate this from, say, some initial value minus 1 over some initial temperature to some temperature where I'm at [? length ?] [? scale ?] l. What I would have on the right hand side would be n minus 2 over [? 2yl ?], OK?

So I start very, very close to the origin T equals 0. So I have a very strong coupling at the beginning. So this factor is huge. T is [? more than ?] 1 over T is huge. I have a huge coupling. And then I rescale to a point where the coupling has become weak, let's say some number of order of 1, order of 1 or order of 0. In any case, it is overwhelmingly smaller than this.

How far did I have to go? I had to rescale by a factor of l that is related to the temperature that I started with by this factor, except that I forgot the minus that I had in front of the whole thing. So the resulting l will be large and positive.

And the correlation length-- the length scale at which we arrived at the coupling, which is of the order of 1 or 0, is whatever my initial length scale was times this factor that I have rescaled by, b, which is e to the l. And so this is a exponential of n minus 1 over 2 pi times 1 over T. The statement is that if you're having 2

dimensions, a system of, let's say, 3 component spins-- and that is something that has a lot of experimental realizations-- you find that as you go towards low temperature, the size of domains that are ordered diverges according to this nice universal form.

And let's say around 1995 or so, when people had these high temperature superconductors which are effectively 2 dimensional layers of magnets-- they're actually antiferromagnets, but they are still described by this [INAUDIBLE] with n equals 3-- there were lots of x-ray studies of what happens to the ordering of these antiferromagnetic copper oxide layers as you go to low temperatures. And this form was very much used and confirmed.

OK, so that's really one thing that one can get from this analysis that has been explicitly [? confirmed ?] [? for ?] experiments. And finally, there's one case in this that I have not mentioned so far, which is n equals 2. And when I am at n equals 2, what I have is that the first and second order terms in this series are both vanishing. And I really at this stage don't quite know what is happening.

But we can think about it a little bit. And you can see that if you are n equals 2, then essentially, you have a 1 component angle. And if I write the theory in terms of the angle theta, let's say, between neighboring spins, then the expansions would simply be gradient of theta squared. And there isn't any other mode to couple with.

You may worry a little bit about gradient of theta to the 4th and such things. But a little bit of thinking will convince you that all of those terms are irrelevant. So as far as we can show, there is reason that essentially this series for n equals 2 is 0 at all orders, which means that as far as this analysis is concerned, there is a kind of a line of fixed points. You start with any temperature, and you will stay at that temperature, OK?

Still you would say that even if you have this gradient of theta squared type of theory, the fluctuations that you have are solutions of 1 over q squared. And the integral of 1 over 2 squared in 2 dimensions is logarithmically divergent. So the more correct statement of the Mermin-Wagner's theorem is that there should be no

long range order in d less than or equal to 2.

Because for d equals 2 also, you have these logarithmic divergence of fluctuations. So you may have thought that you are pointing along, say, the y direction. But you average more and more, and you see that the extent of the fluctuations in angle are growing logarithmically.

You say that once that logarithm becomes of the order of pi, I have no idea where my angle is. There should be no true long range order. And I'm not going to try to interpret this too much. I just say that Mermin-Wagner's theorem says that there should be no true long range order in systems that have continuous symmetry in 2 dimensions and below, OK?

And that statement is correct, except that around that same time, Stanley and Kaplan did low temperature series analysis-- actually, no, I'm incorrect-- did high temperatures series of these spin models in 2 dimensions. And what they found was, OK, let's [? re-plot ?] susceptibility as a function of temperature. We calculate our best estimate of susceptibility from the high temperature series. And what they do is, let's say, they look at the system that corresponds to n equals 3.

And they see that the susceptibility diverges only when you get in the vicinity of 0 temperature, which is consistent with all of these statements that first of all, this is a [? direct ?] correlation [? when it ?] only diverges at 0 temperature. And divergence of susceptibility has to be coupled through that. And therefore, really, the only exciting thing is right at 0 temperature, there is no region where this is long range order, except that when they did the analysis for n equals 2, they kept getting signature that there is a phase transition at a finite temperature in d equals 2 for this xy model that described by just an [INAUDIBLE], OK? So there is lots of numerical evidence of phase transition for n equals 2 in d equals [? 2, ?] OK?

So this is another one of those puzzles which [INAUDIBLE] if we interpret the existence of a diverging susceptibility in the way that we are used to, let's say the Ising model and all the models that we have discussed so far, in all cases that we have seen, essentially, the divergence of the susceptibility was an indicator of the

onset of true long range order so that on the other side, you had something like a magnet. But that is rigorously ruled out by Mermin-Wagner.

So the question is, can we have a phase transition in the absence of symmetry breaking? All right? And we already saw one example of that a couple of lectures back when we were doing the dual of the 3-dimensional Ising model.

We saw that the 3-dimensional Ising model, its dual had a phase transition but was rigorously prevented from having true long range order. So there, how did we distinguish the different phases? We found some appropriate correlation function. And we showed that that correlation function had different behaviors at high and low temperature. And these two different behaviors could not be matched. And so the phase transition was an indicator of the switch-over in the behavior of the correlation functions.

So here, let's examine the correlation functions of our model. And the simplest correlation that we can think of for a system that is described by unit spins is to look at the spin at some location and the spin at some far away location and ask how correlated they are to each other? And so basically, there is some kind of, let's say, underlying lattice.

And we pick 2 points at 0 and at r. And we ask, what is the dot product of the spins that we have at these 2 locations? And clearly, this is invariant under the [? global ?] rotation.

What I can do is I can pick some kind of axis and define angles with respect to some axis. Let's say with respect to the x direction, I define an angle theta. And then clearly, this is the expectation value of cosine of theta 0 minus theta r.

Now, this quantity I can asymptotically calculate both at high temperatures and low temperatures and compare them. So let's do a high T expansion. For the high T expansion, I sort of go back to the discrete model and say that what I have here is a system that is characterized by a bunch of angles that I have to integrate in theta i.

I have the cosine of theta 0 minus theta r. And I have a weight that wants to make

near neighbours to be parallel. And so I will write it as product over nearest neighbors, p to the K cosine of theta i minus theta j.

OK, so the dot product of 2 spins I have written as the cosine between [? nearest ?] neighbors. And of course, I have to then divide by [INAUDIBLE]. Now, if I'm doing the high temperature expansion, that means that this coupling constant K scaled by temperature is known. And I can expand this as 1 plus K cosine of theta i minus theta j [? plus ?] [? higher ?] orders in powers of K of course, OK?

Now, this looks to have the same structure as we had for the Ising model. In the Ising model, I had something like sigma i sigma j. And if I had a sigma by itself and I summed over the possible values, I would get 0.

Here, I have something like a cosine of an angle. And if I integrate, let's say, d theta 0 cosine of theta 0 minus something, just because theta 0 can be both positive as [? it just ?] goes over the entire angle, this will give me 0. So this cosine I better get rid of.

And the way that I can do that is let's say I multiply cosine of theta 0 minus theta r with one of the terms that I would get in the expansion, such as, let's say, a factor of K cosine of theta 0 minus theta 1. So if I call the next one theta 1, I will have a term in the expansion that is cosine of theta 0 minus theta 1, OK? Then this will be non-zero because I can certainly change the origin.

I can write this as integral d theta 0 minus theta 1-- I can call phi. This would be cosine of phi from here. This becomes cosine of theta 0 minus theta 1 minus phi. And this I can expand as cosine of a 0 minus theta 1 cosine of phi minus sine of theta 0 minus theta 1 sine of phi.

Then cosine integrated against sine will give me 0. Cosine integrated against cosine will give me 1/2. So this becomes 1/2 cosine of theta 0 minus theta 1 theta r. What did I-- For theta 0, I am writing phi plus theta 1. So this becomes phi plus theta 1 minus theta r. This becomes cosine of theta 1 minus theta r. This becomes theta 1 minus theta r. This becomes theta 1 minus theta r.

OK, so essentially, we had a term that was like a cosine of theta 0 minus theta r from here. Once we integrate over this bond, then I get a factor of 1/2, and it becomes like a connection between these two. And you can see that I can keep doing that and find the path that connects from 0 to r.

For each one of the bonds along this path, I pick one of these factors. And this allows me to get a finite value. And what I find once I do this is that through the lowest order, I have to count the shortest number of paths that I have between the two, K. I will get a factor of K. And then from the averaging over the angles, I will get 1/2. So it would b K over 2 [INAUDIBLE]. By this we indicate the shortest path between the 2, OK?

So the point is that K is a small number. If I go further and further away, this is going to be exponentially small in the distance between the 2 spaces, where [? c ?] can be expressed in something that has to do with K. So this is actually quite a general statement. We've already seen it for the Ising model. We've now seen it for the xy model.

Quite generally, for systems at high temperatures, once can show that correlations decay exponentially in separation because the information about the state of one variable has to travel all the way to influence the other one. And the fidelity by which the information is transmitted is very small at high temperatures. So OK?

So this is something that you should have known. We are getting the answer. But now what happens if I go and look at low temperatures? So for low temperatures, what I need to do is to evaluate something that has to do with the behavior of these angles when I go to low temperatures. And when I go to low temperatures, these angles tend to be very much aligned to each other. And these factors of cosine I can therefore start expanding around 1.

So what I end up having to do is something like a product over i theta i cosine of theta 0 minus theta r. I have a product over neighbors of factors such as K over 2 theta i minus theta j squared, [? as ?] I expand the Gaussian, expand the cosine.

And in the denominator I would have exactly the same thing without this.

So essentially, we see that since the cosine is the real part of e to the i theta 0 minus theta r, what I need to do is to calculate the average of this assuming the Gaussian weight. So the theta is Gaussian distributed, OK? Now-- actually, this [INAUDIBLE] I can take the outside also. It doesn't matter. I have to calculate this expectation value. And this for any Gaussian expectation value of e to the i some Gaussian variable is minus 1/2 the average of whatever you have, weight.

And again, in case you forgot this, just insert the K here. You can see that this is the characteristic function of the Gaussian distributed variable, which is this difference. And the characteristic function I can start expanding in terms of the cumulants. The first cumulant, the average is 0 by symmetry.

So the first thing that will appear, which would be at the order of K squared, is going to be the variance, which is what we have over here. And since it's a Gaussian, all higher order terms in this series [? will. ?] Another way to do is to of course just complete the square. And this is what would come out, OK?

So all I need to do is to calculate the expectation value of this quantity where the thetas are Gaussian distributed. And the best way to do so is to go to Fourier space. So I have integral. For each one of these factors of theta 0 minus theta r, I will do an integral d2 q 2 pi squared. I have 1 minus e to the iq.r, which is from theta 0 minus theta r. And then I have a theta tilde q.

I have two of those factors. I have d2 q prime 2 pi squared. I have e to the minus iq prime dot r theta tilde q prime. And then this average simply becomes this average. And the different modes are independent of each other. So I will get a 2 pi to the d-- actually, 2 here, a delta function q plus q prime.

And for each mode, I will get a factor of Kq squared because after all, thetas are very much like the pi's that I had written at the beginning, OK? So what I will have is that this quantity is integral over one q's. Putting these two factors together, realizing that q prime is minus q will give me 2 minus 2 cosine of q.r divided by Kq

squared.

So there is an overall scale that is set by 1 over K, by temperature. And then there's a function [? on ?] [? from ?], which is the Fourier transform of [? ?] 1 over q squared, which, as usual, we call C. We anticipate this to be like a Coulomb potential. Because if I take a Laplacian of C, you can see that-- forget the q-- the [? Laplacian ?] of C from the cosine, I will get a minus q squared cancels that. I will have [INAUDIBLE] d2q 2 pi squared. Cosine itself will be left. The 1 over q squared disappears.

This is e to the iqr you plus e to the minus iqr over 2. Each one of them gives a delta function. So this is just a delta function. So C is the potential that you have from a unit charge in 2 dimensions. And again, you can perform the usual Gaussian procedure to find that the gradient of C times 2 pi r is the net charge that is enclosed, which is [? unity ?].

So gradient of C, which points in the radial direction is going to be 1 over 2 pi r. And your C is going to be log of r divided by 2 pi. So this is 1 over K log of r divided by 2 pi. And I state that when essentially the 2 angles are as close as some short distance cut-off, fluctuations vanish. So that's how I set the 0 of my integration, OK?

So again, you put that over here. We find that s0.s of r in the low temperature limit is the exponential of minus 1/2 of this. So I have log of r over a divided by 4 pi K. And I will get a over r to the power of 1 over 4 pi K. And I kind of want to check that I didn't lose a factor of 2 somewhere, which I seem to have.

Yeah, I lost a factor of 2 right here. This should be 2 because of this 2, if I'm using this definition. So this should be 2. And this should be 2 pi K. OK.

So what have we established? We have looked as a function of temperature what the behavior of this spin-spin correlation function is. We have established that in the higher temperature limit, the behavior is something that falls off exponentially with separation.

We have also established that at low temperature, it falls off as a power law in

separation, OK? So these two functional behaviors are different. There is no way that you can connect one to the other. So you pick two spins that are sufficiently far apart and then move the separation further and further away. And the functional form of the correlations is either a power-law decay, power-law decay, or an exponential decay.

And in this form, you know you have a high temperature. In this form, you know you have a low temperature. So potentially, there could be a phase transition separating the distinct behaviors of the correlation function. And that could potentially be underlying what is observed over here, OK? Yes?

AUDIENCE: So where could we make the assumption we're at a low temperature in the second expansion?

PROFESSOR: When we expanded the cosines, right? So what I should really do is to look at the terms such as this. But then I said that I'm low enough temperature so that I look at near neighbors, and they're almost parallel. So the cosine of the angle difference between them is the square of that small angle.

AUDIENCE: Thank you.

PROFESSOR: Yeah. OK? So actually, you may have said that I could have done the same analysis for small angle expansions not only for n equals 2, but also for n equals 3, et cetera. That would be correct. Because I could have made a similar Gaussian analysis for n equals e also. And then I may have concluded the same thing, except that I cannot conclude the same thing because of this thing that we derived over here.

What this shows is that the expansion around 0 temperature regarded as Gaussian is going to break down because of the non-linear coupling that we have between modes. So although I may be tempted to write something like this for n equals 3, I know why it is wrong. And I know the correlation length at which this kind of behavior will need to be replaced with this type of behavior because effectively, the expansion parameter became of the order of 1.

But I cannot do that for the xy model. I don't have similar reason. So then the

question becomes, well, how does this expansion then eventually break down so that I will have a phase transition to a phase where the correlations are decaying exponentially?

And you may say, well, I mean, it's really something to do with having to go to higher and higher ordered terms in the expansion of the cosine. And it's going to be something which would be very difficult to figure out, except that it turns out that there is a much more elegant solution. And that was proposed by [? Kastelitz ?] and [? Thales. ?]

And they said that what you have left out in the Gaussian analysis are topological defects, OK? That is, when I did the expansion of the cosine and we replaced the cosine with the difference of the angles squared, that's more or less fine, except that I should also realize that cosine maintains its value if the angle difference goes up by 2 pi.

And you say, well, neighboring spins are never going to be 2 pi different or pi different because they're very strongly coupled. Does it make any difference? Turns out that, OK, for the neighboring spins, it doesn't make a difference. But what if you go far away?

So let's imagine that this is, let's say, our system of spins. And what I do is I look at a configuration such as this. Essentially, I have spins [? radiating ?] out from a center such as this, OK?

There is, of course, a lot of energy costs I have put over here. But when I go very much further out, let's say, very far away from this plus sign that I have indicated over here, and I follow what the behavior of the spins are, you see that as I go along this circuit, the spins start by pointing this way, they go point this way, this way, et cetera. And by the time I carry a circuit such as this, I find that the angle theta has also rotated by 2 pi, OK?

Now, this is clearly a configuration that is going to be costly. We'll calculate its cost. But the point is that there is no continuous deformation that you can make that will

map this into what we were expanding around here with all of the cosines being parallel to each other. So this is a topologically distinct contribution from the Gaussian one, the Gaussian term that we've calculated.

Since the direction of the rotation of the spins is the same as the direction of the circuit in this case, this is called a plus topological defect. There is a corresponding minus defect which is something like this. OK, and for this, you can convince yourself that as you make a circuit such as this that the direction of the arrow actually goes in the opposite direction, OK? This is called a negative sign topological defect.

Now, I said, well, let's figure out what the energy cost of one of these things is. If I'm away from the center of one of these defects, then the change in angle is small because the change in angle if I go all the way around a circle of radius r should come back to be 2 pi. 2 pi is the uncertainty that I have from the cosines.

So what I have is that the gradient of theta [? between ?] neighboring angles times 2 pi r, which is this radius, is 2 pi. And this thing, here it is plus 1. Here it is minus 1. And in general, you can imagine possibilities where this is some integer that is like 2 or minus 2 or something else that is allowed by this degeneracy of this cosine, OK?

So you can see that when you are far away from the center of whatever this defect is, the gradient of theta has magnitude that is n over r, OK? And as you go further and further, it becomes smaller and smaller. And the energy cost out here you can obtain by essentially expanding the cosine is going to be proportional to the change in angle squared.

So the cost of the defect is an integral of 2 pi rdr times this quantity n over r squared multiplied by the coefficient of the expansion of the cosine, or K over 2. And this integration I have to go all the way to ends up of my system. Let's call it l.

And then I can bring it down, not necessary to the scale of the lattice spacing, but maybe to scale of 5 lattice spacing or something like this, where the approximations that I have used of treating this as a continuum are still valid. So I will pick some

kind of a short distance cut-off a. And then whatever energy is at scales that are below a, I will add to a core energy that depends on whatever this a is. I don't know what that is.

So basically, there is some core energy depending on where I stop this. And the reason that this is more important-- because here I have an integral of 1 over r. And an integral of 1 over r is something that is logarithmically divergent. So I will get K. I have 2 cancels the 2. I have pi en squared log of l over a, OK?

So you can see that creating one of these defects is hugely expensive. And energy that as your system becomes bigger and bigger and we're thinking about infinite sized systems is logarithmically large. So you would say these things will never occur because they cost an infinite amount of energy.

Well, the thing is that entropy is also important. So if I were to calculate the partition function that I would assign to one of these defects, part of it would be exponential of this energy. So I would have this e to the minus this core energy. I would have this exponential of K pi n squared log of l over a. So that's the [INAUDIBLE] weight for this.

But then I realize that I can put this anywhere on the lattice so there is an entropy [? gain ?] factor. And since I have assigned this to have some kind of a bulk to it as some characteristic size a, the number of distinct places that I can put it is over the order of l over a squared. So basically, I take my huge lattice and I partition it into sizes of a's. And I say I can put it in any one of these configurations.

You can see that the whole thing is going to be e to the minus this core energy. And then I have l over a to the power of 2 minus pi K n squared, OK? So the logarithmic energy cost is the same form as the logarithmic entropy gain that you have over here. And this precise balance will give you a value of K such that if K is larger than 2 over pi n squared, this is going to be an exponentially large cost. There is a huge negative power of l here that says, no, you don't want to create this.

But if K becomes weak such that the 2, the entropy factor, [? wins ?], then you will

start creating [INAUDIBLE]. So you can see that over here, suddenly we have a mechanism along our picture over here, maybe something like K, which is 2 over pi, such that on one side you would say, I will not have topological defect and I can use the Gaussian model. And on the other side, you say that I will spontaneously create these topological defects. And then the Gaussian description is no longer valid because I have to now really take care of the angular nature of these variables, OK?

So this is a nice picture, which is only a zeroed order picture. And to zeroed order, it is correct. But this is not fully correct. Because even at low temperatures, you can certainly create hairs of plus minus defects, OK?

And whereas the field for one of them will fall off at large distances, the gradient of theta is 1 over r, if you superimpose what is happening for two of these, what you will convince yourself that if you have a pair of defects of opposite sign at the distance d that the distortion that they generate at large distances falls off not the as 1 over r, which is, if you like, a monopole field, but as d over r squared, which is a dipole field. So whenever you have a dipole, you will have to multiply by the separation of the charges in the dipole. And that is compensated by a factor of 1 over r in the denominator. There is some angular dependence, but we are not so interested in that.

Now, if I were to integrate this square, we can see that it is something that is convergent at large distances. And so this is going to be finite. It is not going to diverge as the size of the system, which means that whereas individual defects there was no way that I could create in my system, I can always create pairs of these defects.

So the correct picture that we should have is not that at low temperatures you don't have defects, at high temperatures you have these defects spontaneously appearing. The correct picture is that at low temperatures what you have is lots and lots of dipoles that are pretty much bound to each other. And when you go to high temperatures, what happens is that you will have these pluses and minuses

unbound from each other.

So if you like, the transition is between molecules to a plasma as temperature is changed. Or if you like, it is between an insulator and a conductor. And how to mathematically describe this phase transition in 2 dimensions, which we can rigorously, will be what we will do in the next couple of lectures.