Review Problems

The test is 'closed book,' but if you wish you may bring a one-sided sheet of formulas. The intent of this sheet is as a reminder of important formulas and definitions, and not as a compact transcription of the answers provided here. If this privilege is abused, it will be revoked for future tests. The test will be composed entirely from a subset of the following problems, **as well as those in problem sets 3 and 4**. Thus if you are familiar and comfortable with these problems, there will be no surprises!

****

1. *Scaling in fluids:* Near the liquid–gas critical point, the free energy is assumed to take the scaling form $F/N = t^{2-\alpha} g(\delta\rho/t^\beta)$, where $t = |T - T_c|/T_c$ is the reduced temperature, and $\delta\rho = \rho - \rho_c$ measures deviations from the critical point density. The leading singular behavior of any thermodynamic parameter $Q(t, \delta\rho)$ is of the form t^x on approaching the critical point along the isochore $\rho = \rho_c$; or $\delta\rho^y$ for a path along the isotherm $T = T_c$. Find the exponents x and y for the following quantities:

(a) The internal energy per particle $\langle H\rangle/N$, and the entropy per particle $s = S/N$.

(b) The heat capacities $C_V = T\partial s/\partial T\,|_V$, and $C_P = T\partial s/\partial T\,|_P$.

(c) The isothermal compressibility $\kappa_T = \partial\rho/\partial P\,|_T/\rho$, and the thermal expansion coefficient $\alpha = \partial V/\partial T\,|_P/V$.

Check that your results for parts (b) and (c) are consistent with the thermodynamic identity $C_P - C_V = TV\alpha^2/\kappa_T$.

(d) Sketch the behavior of the latent heat per particle L, on the coexistence curve for $T < T_c$, and find its singularity as a function of t.

****

2. *The Ising model:* The differential recursion relations for temperature T, and magnetic field h, of the Ising model in $d = 1 + \epsilon$ dimensions are

$$\begin{cases} \dfrac{dT}{d\ell} = -\,\epsilon T + \dfrac{T^2}{2} \quad , \\ \dfrac{dh}{d\ell} = dh \quad . \end{cases}$$

(a) Sketch the renormalization group flows in the (T, h) plane (for $\epsilon > 0$), marking the fixed points along the $h = 0$ axis.

(b) Calculate the eigenvalues y_t and y_h, at the critical fixed point, to order of ϵ.

(c) Starting from the relation governing the change of the correlation length ξ under renormalization, show that $\xi(t,h) = t^{-\nu} g_\xi \left(h/|t|^\Delta\right)$ (where $t = T/T_c - 1$), and find the exponents ν and Δ.

(d) Use a hyperscaling relation to find the singular part of the free energy $f_{\text{sing.}}(t,h)$, and hence the heat capacity exponent α.

(e) Find the exponents β and γ for the singular behaviors of the magnetization and susceptibility, respectively.

(f) Starting the relation between susceptibility and correlations of local magnetizations, calculate the exponent η for the critical correlations ($\langle m(\mathbf{0})m(\mathbf{x})\rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$).

(g) How does the correlation length diverge as $T \to 0$ (along $h = 0$) for $d = 1$?

****

3. *Longitudinal susceptibility:* While there is no reason for the longitudinal susceptibility to diverge at the mean-field level, it in fact does so due to fluctuations in dimensions $d < 4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2\right] \quad ,$$

describing an n–component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t < 0$.

(a) Let $\vec{m}(\mathbf{x}) = \left(\overline{m}+\phi_\ell(\mathbf{x})\right)\hat{e}_\ell+\vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\overline{m} \int d^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

(c) Calculate the Gaussian (bare) expectation values $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle_0$ and $\langle\phi_{t,\alpha}(\mathbf{q})\phi_{t,\beta}(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\ \ \vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\rangle_0$ using Wick's theorem. (Don't forget that $\vec{\phi}_t$ is an $(n-1)$ component vector.)

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second-order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non–zero.

(f) Using the form of U in Fourier space, write the correction term as a product of two 4–point expectation values similar to those of part (d). Note that only connected terms for the longitudinal 4–point function should be included.

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

****

4. *Crystal anisotropy:* Consider a ferromagnet with a tetragonal crystal structure. Coupling of the spins to the underlying lattice may destroy their full rotational symmetry. The resulting anisotropies can be described by modifying the Landau–Ginzburg Hamiltonian to

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \frac{r}{2}m_1^2 + v\,m_1^2\,\vec{m}^2\right],$$

where $\vec{m} \equiv (m_1, \cdots, m_n)$, and $\vec{m}^2 = \sum_{i=1}^n m_i^2$ ($d = n = 3$ for magnets in three dimensions). Here $u > 0$, and *to simplify calculations we shall set* $v = 0$ throughout.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the n component magnetization space are selected for $r > 0$, and for $r < 0$?

(b) Using the saddle point approximation, calculate the free energies ($\ln Z$) for phases uniformly magnetized *parallel* and *perpendicular* to direction 1.

(c) Sketch the phase diagram in the (t, r) plane, and indicate the phases (type of order), and the nature of the phase transitions (continuous or discontinuous).

(d) Are there Goldstone modes in the ordered phases?

(e) For $u = 0$, and positive t and r, calculate the unperturbed averages $\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle_0$ and $\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle_0$, where $m_i(\mathbf{q})$ indicates the Fourier transform of $m_i(\mathbf{x})$.

(f) Write the fourth order term $\mathcal{U} \equiv u\int d^d\mathbf{x}(\vec{m}^2)^2$, in terms of the Fourier modes $m_i(\mathbf{q})$.

(g) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle$. (You can leave your answers in the form of some integrals.)

(h) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle$.

(i) Using the above answer, identify the inverse susceptibility χ_{22}^{-1}, and then find the transition point, t_c, from its vanishing to first order in u.

(j) Is the critical behavior different from the isotropic $O(n)$ model in $d < 4$? In RG language, is the parameter r *relevant* at the $\mathcal{O}(n)$ fixed point? In either case indicate the universality classes expected for the transitions.

****

5. *Cubic anisotropy– Mean-field treatment:* Consider the modified Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4 \right] \quad ,$$

for an n–component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term $\sum_{i=1}^{n} m_i^4$, breaks the full rotational symmetry and selects specific directions.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the n component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?

(b) What are the restrictions on u and v for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the (u, v) plane.

(c) In general, higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and ensure stability in the regions not allowed in part (b); (as in case of the tricritical point discussed in earlier problems). With such terms in mind, sketch the saddle point phase diagram in the (t, v) plane for $u > 0$; clearly identifying the phases, and order of the transition lines.

(d) Are there any Goldstone modes in the ordered phases?

****

6. *Cubic anisotropy– ε–expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both u and v show that the recursion relations for these couplings are

$$\begin{cases} \dfrac{du}{d\ell} =\varepsilon u - 4C\left[\ (n+8)u^2 + 6uv\ \right] \\ \dfrac{dv}{d\ell} =\varepsilon v - 4C\left[\ 12uv + 9v^2\ \right] \end{cases} ,$$

where $C = K_d\Lambda^d/(t + K\Lambda^2)^2 \approx K_4/K^2$, is approximately a constant.

(b) Find all fixed points in the (u, v) plane, and draw the flow patterns for $n < 4$ and $n > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

(c) Find the recursion relation for the reduced temperature, t, and calculate the exponent ν at the stable fixed points for $n < 4$ and $n > 4$.

(d) Is the region of stability in the (u, v) plane calculated in part (b) of the previous problem enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative v and $n > 4$?

****

7. *Cumulant method:* Apply the Niemeijer–van Leeuwen first order cumulant expansion to the Ising model on a *square* lattice with $-\beta\mathcal{H} = K\sum_{<ij>} \sigma_i\sigma_j$, by following these steps:

(a) For an RG with $b = 2$, divide the bonds into *intra–cell* components $\beta\mathcal{H}_0$; and *inter–cell* components $\mathcal{U}$.

(b) For each cell α, define a renormalized spin $\sigma'_\alpha = \text{sign}(\sigma^1_\alpha + \sigma^2_\alpha + \sigma^3_\alpha + \sigma^4_\alpha)$. This choice becomes ambiguous for configurations such that $\sum_{i=1}^4 \sigma^i_\alpha = 0$. Distribute the weight of these configurations equally between $\sigma'_\alpha = +1$ and -1 (i.e. put a factor of $1/2$ in addition to the Boltzmann weight). Make a table for all possible configurations of a cell, the internal probability $\exp(-\beta\mathcal{H}_0)$, and the weights contributing to $\sigma'_\alpha = \pm 1$.

(c) Express $\langle\mathcal{U}\rangle_0$ in terms of the cell spins σ'_α; and hence obtain the recursion relation $K'(K)$.

(d) Find the fixed point K^*, and the thermal eigenvalue y_t.

(e) In the presence of a small magnetic field $h\sum_i \sigma_i$, find the recursion relation for h; and calculate the magnetic eigenvalue y_h at the fixed point.

(f) Compare K^*, y_t, and y_h to their exact values.

****

8. *Migdal–Kadanoff method:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, on sites i of a hypercubic lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K\sum_{<ij>} \delta_{s_i,s_j} \ .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Indentify all fixed points and note their stability.

(b) Write down the recursion relation $K'(K)$ in d–dimensions for $b = 2$, using the Migdal–Kadanoff bond moving scheme.

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non–trivial fixed point at finite K^* for $d > 1$.

(d) For $d = 2$, obtain K^* and y_t, for $q = 3$, 1, and 0.

****

9. *The Potts model:* The *transfer matrix* procedure can be extended to Potts model, where the spin s_i on each site takes q values $s_i = (1, 2, \cdots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K\sum_{i=1}^N \delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a q^{th} order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

(b) Calculate the free energy per site.

(c) Give the expression for the correlation length ξ (you don't need to provide a detailed derivation), and discuss its behavior as $T = 1/K \to 0$.

****