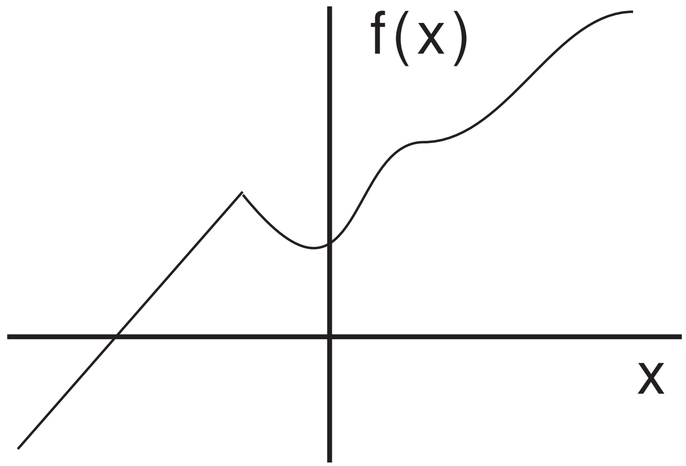
Functions of a random variable

Given: $p_x(\zeta)$ and $f(x)$

Find: $p_f(\eta)$



A. Sketch $f(x)$. Find where $f(x) < \eta$

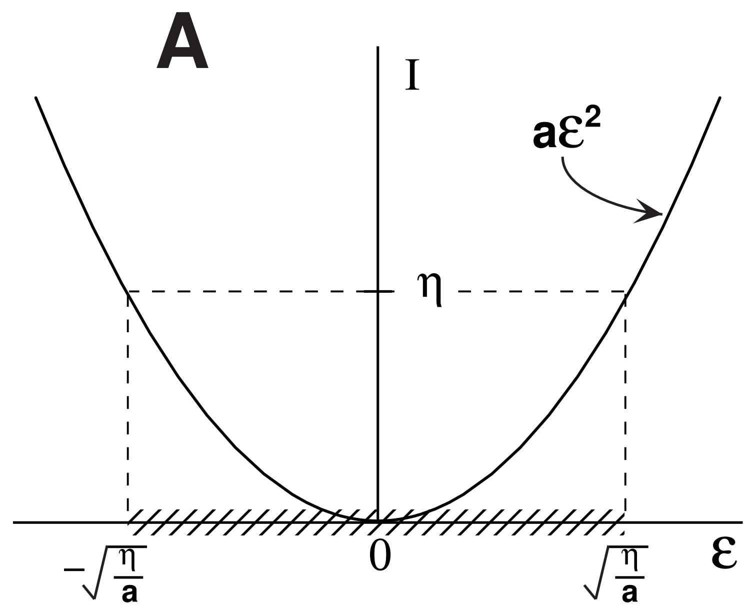
B. Integrate to find $P_f(\eta)$.

C. Differentiate to find $p_f(\eta)$.

Example Intensity of light

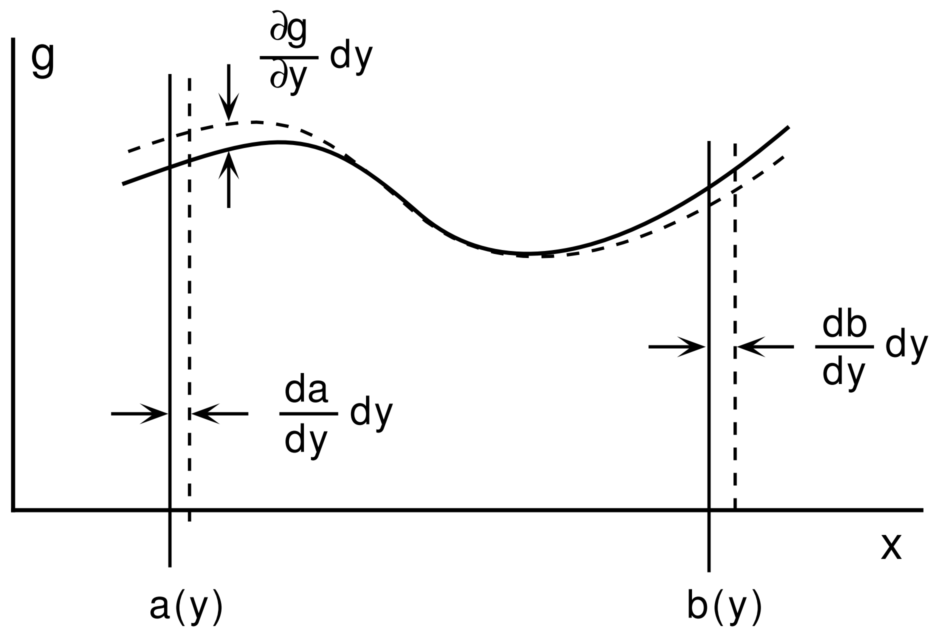
$$I = a\mathcal{E}^2$$

$$p(\mathcal{E}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\mathcal{E}^2/2\sigma^2]$$



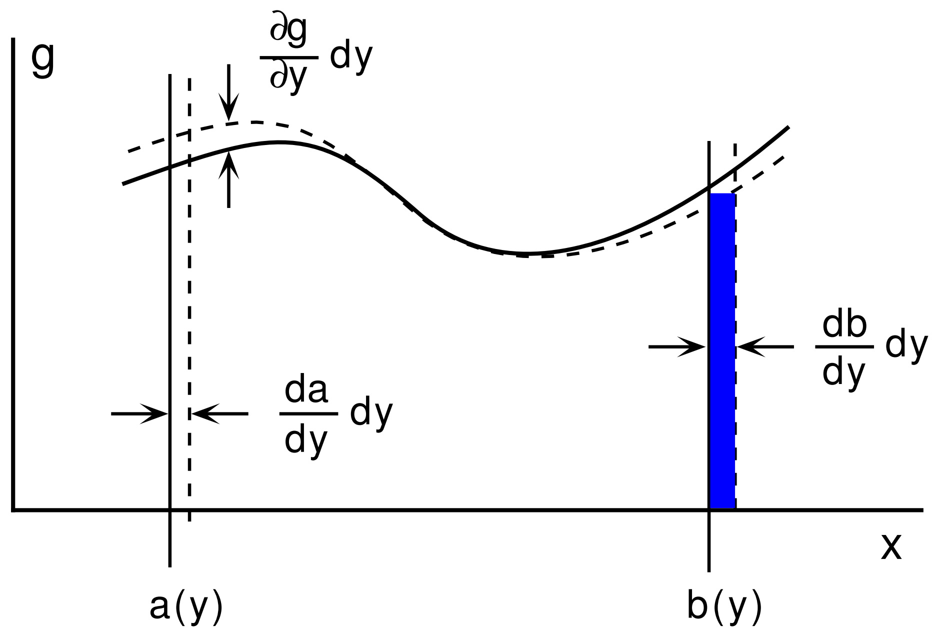
B

$$P_I(\eta) = \int_{-\sqrt{\eta/a}}^{\sqrt{\eta/a}} p_{\mathcal{E}}(\zeta)\, d\zeta$$



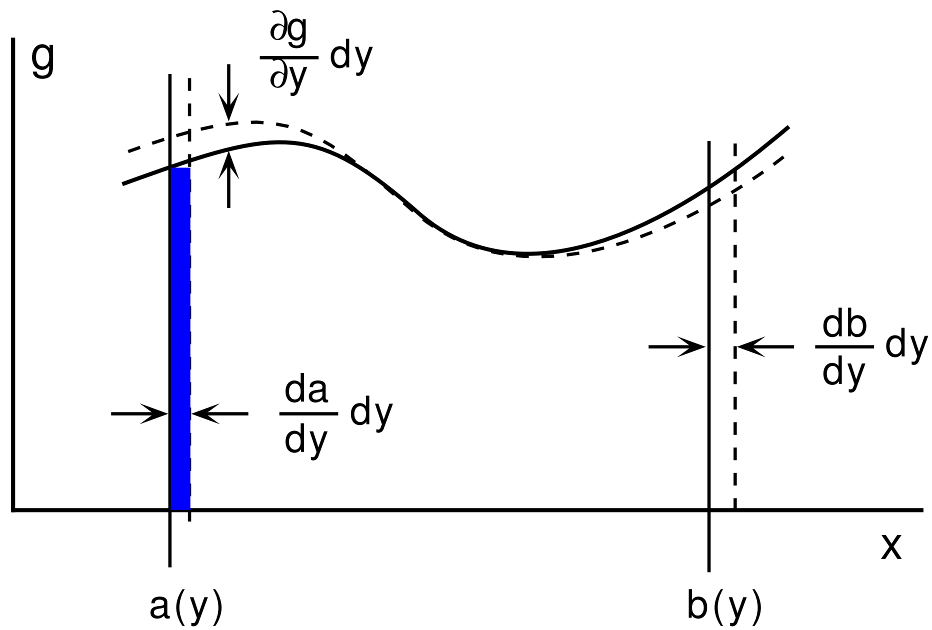
$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(y,x)\,dx =$$

$$g(y,x=b(y))\frac{db(y)}{dy} - g(y,x=a(y))\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y,x)}{\partial y}\,dx$$



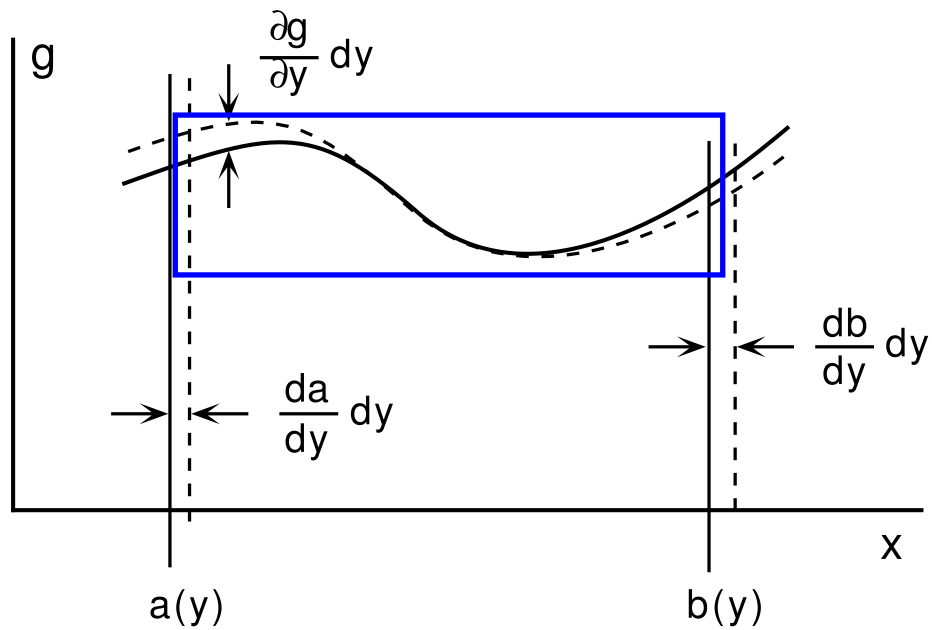
$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(y,x)\,dx =$$

$$\boxed{g(y,x=b(y))\frac{db(y)}{dy}} - g(y,x=a(y))\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y,x)}{\partial y}\,dx$$



$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(y,x)\,dx =$$

$$g(y,x=b(y))\frac{db(y)}{dy} \boxed{- g(y,x=a(y))\frac{da(y)}{dy}} + \int_{a(y)}^{b(y)} \frac{\partial g(y,x)}{\partial y}\,dx$$



$$\frac{d}{dy}\int_{a(y)}^{b(y)} g(y,x)\,dx =$$

$$g(y,x=b(y))\frac{db(y)}{dy} - g(y,x=a(y))\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y,x)}{\partial y}\,dx$$

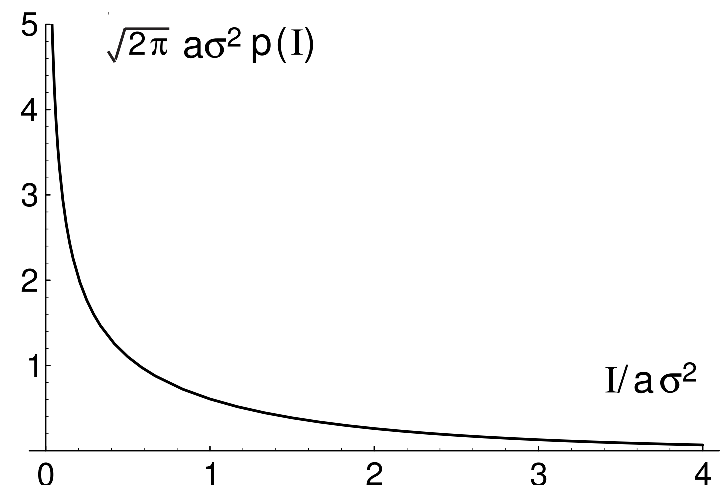
C In general

$$p_I(\eta) \;=\; \frac{1}{2}\frac{1}{\sqrt{\eta a}}\, p_{\mathcal{E}}(\sqrt{\eta/a}) - (-\frac{1}{2}\frac{1}{\sqrt{\eta a}})\, p_{\mathcal{E}}(-\sqrt{\eta/a})$$

$$= \frac{1}{2}\frac{1}{\sqrt{\eta a}}\,[p_{\mathcal{E}}(\sqrt{\eta/a}) + p_{\mathcal{E}}(-\sqrt{\eta/a})]$$

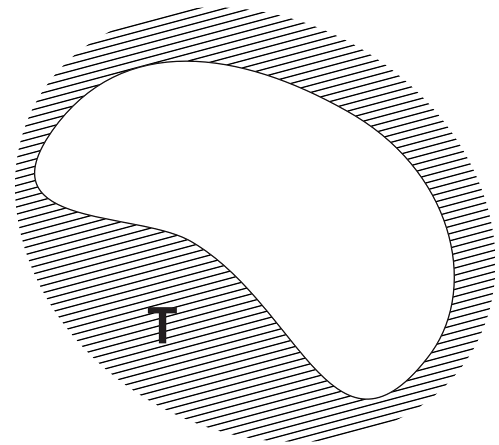
In our particular case

$$p(I) = \frac{1}{\sqrt{2\pi a\sigma^2 I}}\exp[-\frac{I}{2a\sigma^2}]$$

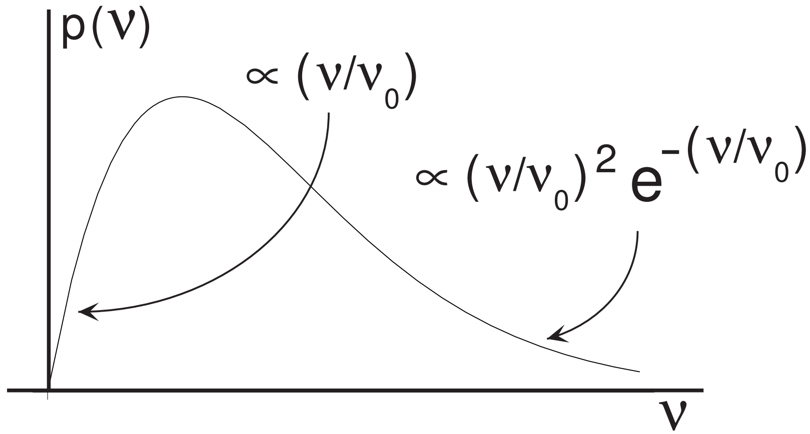


Example Black Body Radiation

$$p(\nu) = \underbrace{\frac{1}{2\zeta(3)}}_{1/2.404} \frac{1}{\nu_0} \frac{(\nu/\nu_0)^2}{\exp[\nu/\nu_0] - 1}$$



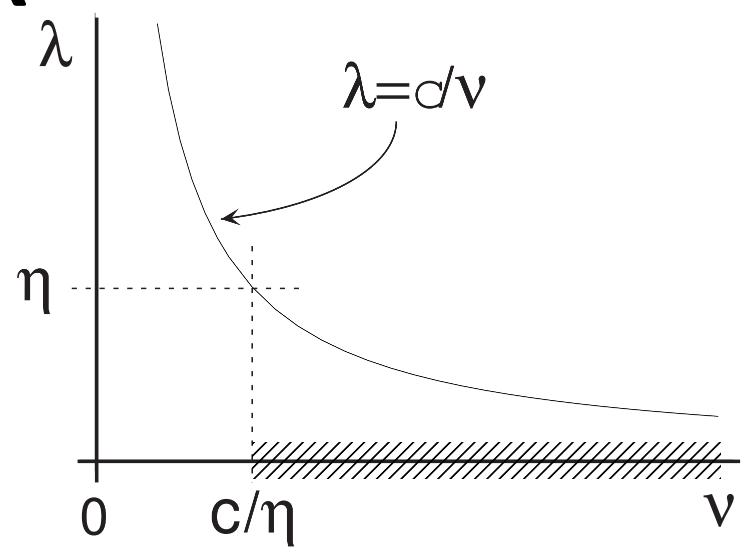
$$\nu_0 = kT/h$$



Given $\lambda = c/\nu$ and $p(\nu)$

Find $p(\lambda)$

A



B

$$P_\lambda(\eta) = \int_{c/\eta}^{\infty} p_\nu(\zeta)\, d\zeta$$

C

In general

$$p_\lambda(\eta) = -(-c/\eta^2)\, p_\nu(c/\eta)$$

In our case

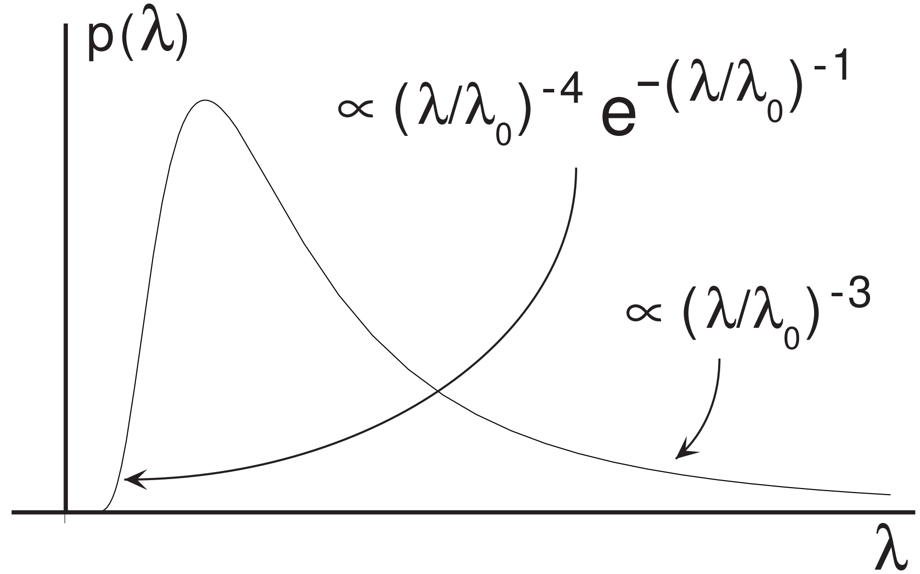
$$p_\lambda(\eta) = \frac{c}{\eta^2}\ \frac{1}{2.404}\ \frac{1}{\nu_0}\ \frac{(c/\eta\nu_0)^2}{\exp[(c/\eta\nu_0)] - 1}$$

Let $\lambda_0 \equiv c/\nu_0$, then

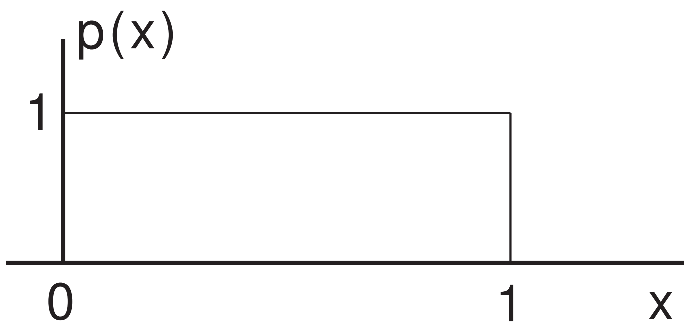
$$p_\lambda(\eta) = \frac{1}{2.404}\ \frac{1}{\lambda_0}\ \left(\frac{\eta}{\lambda_0}\right)^{-4} \frac{1}{\exp[(\eta/\lambda_0)^{-1}] - 1}$$

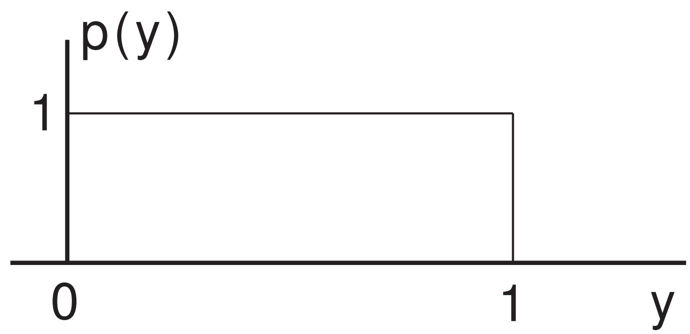
As $(\lambda/\lambda_0)\rightarrow 0$ $\quad \frac{1}{\exp[(\lambda/\lambda_0)^{-1}]-1} \rightarrow e^{-(\lambda/\lambda_0)^{-1}}$

As $(\lambda/\lambda_0)\rightarrow \infty$ $\quad \frac{1}{\exp[(\lambda/\lambda_0)^{-1}]-1} \rightarrow \frac{1}{(1+(\lambda/\lambda_0)^{-1}-1)} \rightarrow (\lambda/\lambda_0)$



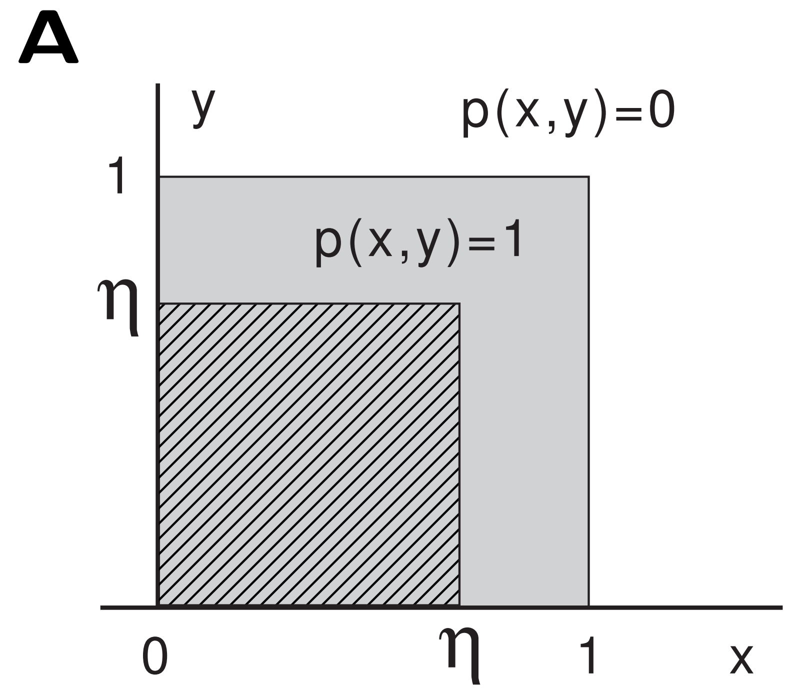
Example Random number generator for programmers





x and y are statistically independent

$z \equiv \text{MAX}(x, y)$ Find $p(z)$

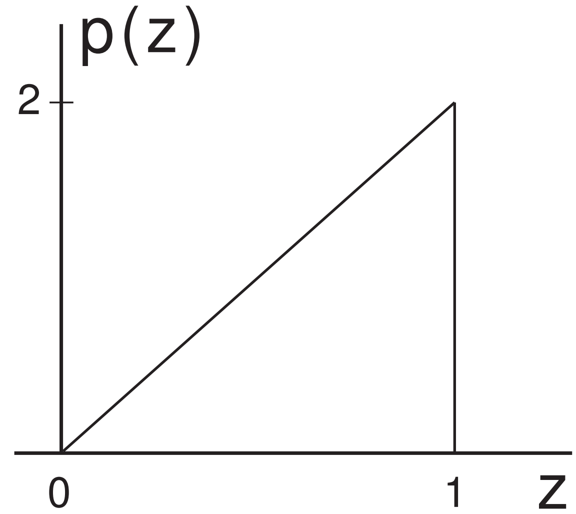


$p(x, y) = p(x)\, p(y)$

Where is $\text{MAX}(x, y) = \eta$?

Where is $\text{MAX}(x, y) < \eta$?

B $P_z(\eta) = \eta^2$



C $p_z(\eta) = 2\eta \quad 0 \leq \eta \leq 1$

$$< z > \;=\; \int_0^1 2\eta^2\, d\eta = (2/3) \left[_0^1 \eta^3 = 2/3\right.$$

$$< z^2 > \;=\; \int_0^1 2\eta^3\, d\eta = (2/4) \left[_0^1 \eta^4 = 1/2\right.$$

$$\text{Var}(z) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}, \qquad \text{S.D.} = \frac{1}{\sqrt{18}} = 0.24$$

Example Desorbing atom

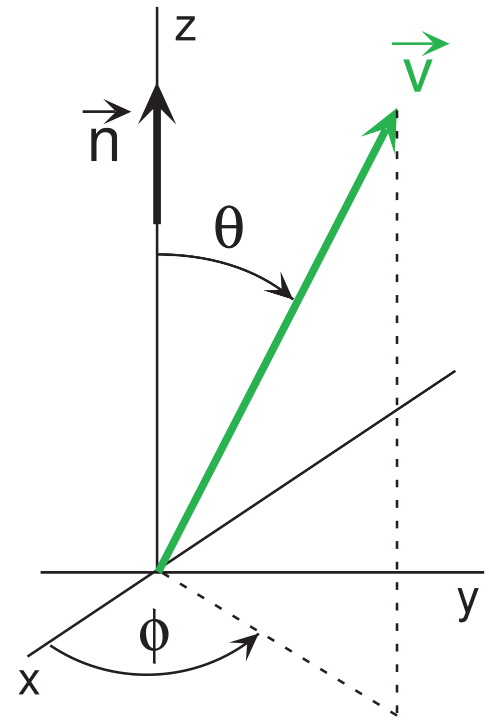
$$p(v,\theta,\phi) = p(v)\,p(\theta)\,p(\phi)$$

$$p(v) = (1/2\sigma^4)\,v^3 \exp[-v^2/2\sigma^2]$$

$$p(\theta) = 2\sin\theta\,\cos\theta$$

$$p(\phi) = 1/2\pi$$

Find $p(v_z)$

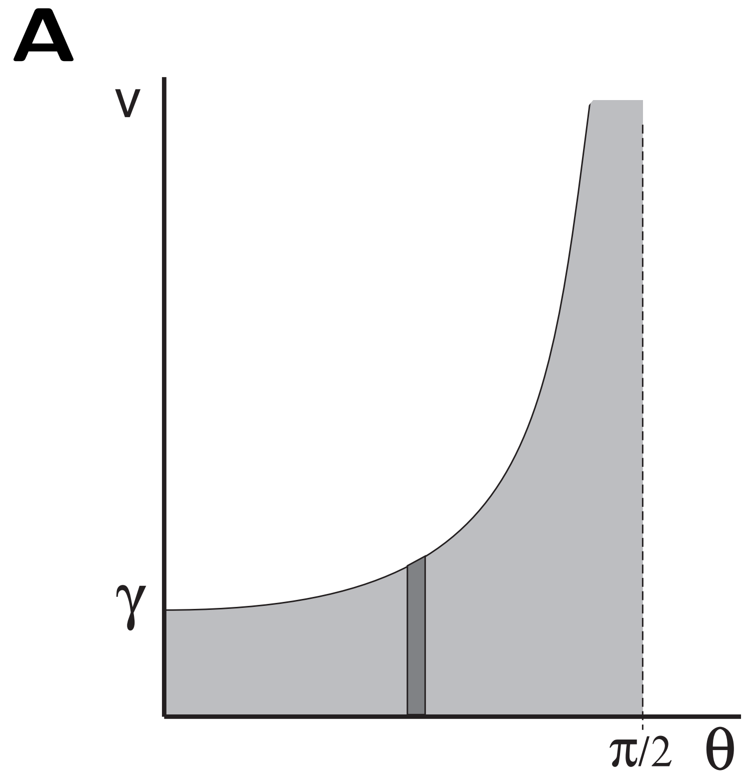


leaving the surface

$$v_z = v \cos\theta$$

$$v \cos\theta < \gamma$$

$$\Rightarrow v < \gamma / \cos\theta$$



B

$$P_{v_z}(\gamma) = \int_0^{\pi/2} \int_0^{\gamma/\cos\eta} p_v(\zeta)\, p_\theta(\eta)\, d\zeta\, d\eta$$

$$= \int_0^{\pi/2} p_\theta(\eta) \left[\int_0^{\gamma/\cos\eta} p_v(\zeta)\, d\zeta \right] d\eta$$

C

$$p_{v_z}(\gamma) = \frac{dP_{v_z}(\gamma)}{d\gamma} = \int_0^{\pi/2} p_\theta(\eta) \left[\tfrac{1}{\cos\eta}\, p_v(\tfrac{\gamma}{\cos\eta}) \right] d\eta$$

$p_{v_z}(\gamma) =$

$$\int_0^{\pi/2} (2 \sin\eta \cos\eta) \left[\frac{1}{\cos\eta} \frac{1}{2\sigma^4} \left(\frac{\gamma}{\cos\eta} \right)^3 \exp[-\frac{1}{2\sigma^2} \frac{\gamma^2}{\cos^2\eta}] \right] d\eta$$

Let $\frac{1}{2\sigma^2} \frac{\gamma^2}{\cos^2\eta} \equiv \mathrm{X}$

$$d\mathrm{X} = -\frac{1}{\sigma^2} \frac{\gamma^2}{\cos^3\eta} (-\sin\eta) d\eta$$

$\eta = 0 \quad \Rightarrow \quad \mathrm{X} = \gamma^2/2\sigma^2;$ $\qquad \eta = \pi/2 \quad \Rightarrow \quad \mathrm{X} = \infty$

$$p_{v_z}(\gamma) \;=\; \frac{\gamma}{\sigma^2}\int_{\gamma^2/2\sigma^2}^{\infty} e^{-X}\, dX = -\frac{\gamma}{\sigma^2}\Big[_{\gamma^2/2\sigma^2}^{\infty}\;\; e^{-X}$$

$$=\; \frac{\gamma}{\sigma^2}\,\mathrm{exp}[-\gamma^2/2\sigma^2] \qquad \gamma > 0$$