PART 2

OCEAN ENGINEERING

HYDROSTATICS

13.00 LECTURE NOTES — OCEAN ENGINEERING HYDROSTATICS

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1 UNITS AND CONVERSIONS

Unfortunately, both metric and English units are commonly used in the design, construction and operation of ships and offshore platforms. While the trend towards the metric (SI) system is growing worldwide, the United States is still locked into the English system (except in technical journals). It is therefore best for you to be familiar with both systems. For your convenience, this section defines the more common quantities, and their conversion factors.

length The length unit is the meter in the SI system, and the foot in the English system. One foot is 0.3048 meters (exactly). The reciprocal value, 3.28, is not exact. If you are fussy, convert meters to feet by dividing by 0.3048. In the English system, there is also the inch (1/12 of a foot), which commonly appears in the measurement of pressure ($pounds/in^2$), and is used to denote an infinitesimally small quantity, such as *tons per inch immersion* which we will run into later.

mass The relevant SI mass units are the kilogram and the tonne (which is 1000 kilograms). While there are mass units in the English system (such as slugs or pounds mass) it is safer not to use them.

force In the SI system, the basic unit of force is the Newton, which is defined as the force acting on one kilogram of mass when subjected to an acceleration of one meter per second squared. The weight of one kilogram mass under standard gravity $g = 9.80665 m/sec^2$ is therefore 9.80665 Newtons. Similarly, the force acting on a tonne mass is 9,806.65 Newtons, which is defined as the metric ton, or tonnef. The preferred SI unit for heavyweights is the mega-Newton (MN) which is 10^6 Newtons. However, the metric tonnef is much more common in the real world. The corresponding units in the English system are the pound force, which is 4.44822 Newtons, and the ton, which is defined as 2240 pounds. The metric tonnef is therefore $9,806.65 \times 4.44822 = 2204.62$ pounds. Thus, the metric tonnef and the English ton are almost equal— one English ton equals 1.016 metric tonnef. Finally, the standard acceleration of gravity in the English system is $9.80665/0.3048 = 32.174\ ft/sec^2$.

2 WHAT IS SEA WATER?

In ocean engineering hydrostatics, we will study the static interaction of a fluid (*sea water*) with engineering objects floating on or underneath the sea surface. It is therefore logical to begin with a discussion of the relevant properties of the fluid medium.

Most of the world consists of fluids and solids. How can we tell which is which? An obvious distinction is that fluids have no distinct shape, but will adapt to the shape of a container in which they are placed. On the other hand, solids retain (more or less) their original shape, even if moved around or placed in a different container. The reason for this difference is that a fluid at rest cannot sustain a shear stress, while a solid can. A fluid can sustain shear stresses by means of relative motion. Thus a sphere of fluid placed in a rectangular container will undergo rapid deformations until it conforms to the shape of the container, and equilibrium is restored without the presence of any shear stresses.

What are shear stresses? Stresses, by definition, are forces per unit area. The force acting on one surface of a small element of a fluid or solid can be decomposed into a normal stress, and two components of stress tangent to the surface, as shown in figure 1. The tangential components of the total stress are termed shear stresses, while the normal component is termed pressure.

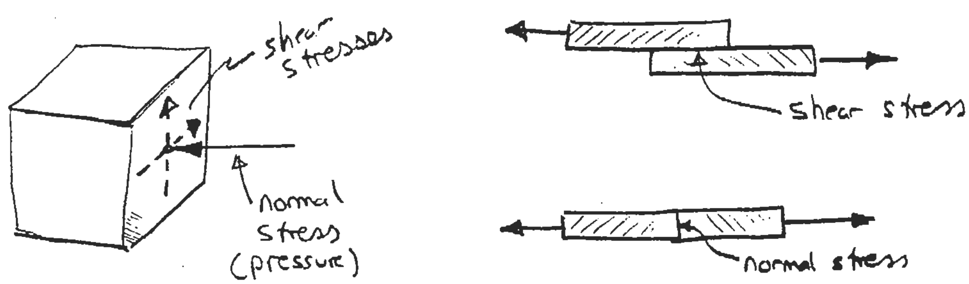


Figure 1: Definition of normal and shear stresses

Fluids can be further subdivided into two categories: liquids and gasses. Here we go again with our handy box. If you put a liquid in a box, it fills up the lower part, up to some level. If you put a gas in a box, it fills it all up.

What makes solids, liquids and gasses different can be explained on the basis of the molecular behavior of each substance. However, for many engineering purposes, it is sufficient to accept the very obvious differences between these substances as given.

Speaking of molecules, however, brings up the concept of a continuum. In the treatment of both hydrostatics and hydrodynamics, one frequently talks about fluid properties *at a point* or talks about a *fluid particle*. Strictly speaking, if we look at a sufficiently small region of a fluid, we will see a bunch of atomic particles buzzing around in a random fashion. Hence, a statement of a fluid property *at a point* really means the average property within a region which is negligibly small compared to the scale of the engineering problem at hand, but infinitely large compared to the molecular scale. This idealization is called a *continuum* representation of the material. One hardly needs to apologize for introducing this idealization, since the two length scales are many orders of magnitude apart.

One important property of a fluid is its density, which is defined as its mass per unit volume and given the symbol ρ. The density of a fluid depends on its temperature, pressure and on the presence of impurities. The latter is particularly significant for sea water, where the presence of salts increases its density by approximately 2.6%. This number, of course, depends on the salinity of the water, but the figure given is accepted as characterizing "standard sea water" in the absence of other information.

Table 2 gives the variation of density with temperature for fresh water and sea water at atmospheric pressure.

The mass density of fresh water, at standard conditions, is $1000 kg/m^3$ or one $tonne/m^3$. I tried converting the English unit value at 59 degrees using the constants provided in the first section, and got $999.008/kg/m^3$, which is close enough.

The density of sea water also depends on pressure. While water is generally idealised as an ***incompressible*** fluid, you can compress anything if you push hard enough. The measure of compressibility is the ***bulk modulus***, K, which is defined as

$$K = -\frac{\Delta p}{\left(\frac{\delta\nabla}{\nabla}\right)} \tag{1}$$

where ∇ is an element of volume of the fluid. The standard value for sea water is $K = 339,000$ pounds per square inch. Thus, if an element of volume is subjected to a pressure of 9,000 pounds per square inch, its volume will be reduced by

$$\frac{\delta\nabla}{\nabla} = \frac{9,000}{339,000} = 0.026 \tag{2}$$

or around 2.6 percent. This does not seem like much, but we will see that this can disturb the delicate balance between weight and buoyancy of a deeply submerged vehicle.

Another consequence of the compressibility of a medium is that acoustic waves will propogate in it. It can be shown (obviously not within the scope of 13.00!) that the speed of sound in a medium is

$$c = \sqrt{\frac{K}{\rho}} \tag{3}$$

which for standard conditions for sea water comes out to be $4,954$ feet per second. Thus, sound travels much faster in water than in air—a fact which is well known to whales and to humans that design sonars.

Table of Density of Water at Atmospheric Pressure

Density of Fresh Water ρ 1b x sec^2/ft^4	Temperature degree F	Density of Sea Water ρ_δ 1b x sec^2/ft^4	Density of Fresh Water ρ 1b x sec^2/ft^4	Temperature degree F	Density of Sea Water ρ_δ 1b x sec^2/ft^4
1.9399	32	1.9947	1.9381	61	1.9901
1.9399	33	1.9946	1.9379	62	1.9898
1.9400	34	1.9946	1.9377	63	1.9895
1.9400	35	1.9945	1.9375	64	1.9893
1.9401	36	1.9944	1.9373	65	1.9890
1.9401	37	1.9943	1.9371	66	1.9888
1.9401	38	1.9942	1.9369	67	1.9885
1.9401	39	1.9941	1.9367	68	1.9882
1.9401	40	1.9940	1.9365	69	1.9879
1.9401	41	1.9939	1.9362	70	1.9876
1.9401	42	1.9937	1.9360	71	1.9873
1.9401	43	1.9936	1.9358	72	1.9870
1.9400	44	1.9934	1.9355	73	1.9867
1.9400	45	1.9933	1.9352	74	1.9864
1.9399	46	1.9931	1.9350	75	1.9861
1.9398	47	1.9930	1.9347	76	1.9858
1.9398	48	1.9928	1.9344	77	1.9854
1.9397	49	1.9926	1.9342	78	1.9851
1.9396	50	1.9924	1.9339	79	1.9848
1.9395	51	1.9923	1.9336	80	1.9844
1.9394	52	1.9921	1.9333	81	1.9841
1.9393	53	1.9919	1.9330	82	1.9837
1.9392	54	1.9917	1.9327	83	1.9834
1.9390	55	1.9914	1.9324	84	1.9830
1.9389	56	1.9912	1.9321	85	1.9827
1.9387	57	1.9910	1.9317	86	1.9823
1.9386	58	1.9908			
1.9384	59	1.9905			
1.9383	60	1.9903			

Figure by MIT OCW.

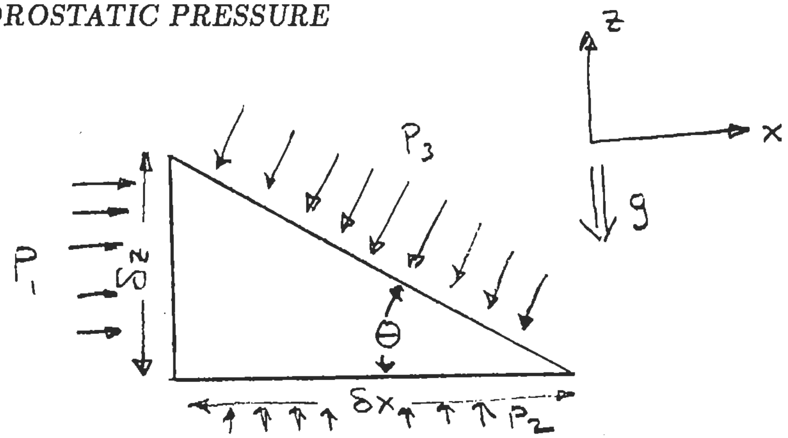


Figure 3: Element of fluid to illustrate omnidirectional property of hydrostatic pressure

Finally, we have to define what is above sea water. Unless we are in a cave, it is air, at a standard sea-level atmospheric pressure of $p_a = 14.7$ pounds per square inch.

3 HYDROSTATIC PRESSURE

We will first establish the fact that hydrostatic pressure at a given point in a fluid is omnidirectional. That is, if we look at a surface of an infinitesimal object immersed in a fluid at rest, the pressure acting on that surface is independent of the orientation of the surface. We can show this easily by taking a triangular prism of fluid of width δx, height δz and depth b as shown in figure 3. The axis are chosen so that gravity points in the $-z$ direction. Let the pressure on the left face be p_1, the pressure on the bottom face p_2 and the pressure on the hypotenuse p_3.

Summing the forces in the x direction:

$$F_x = p_1 b \delta z - p_3 b \sqrt{(\delta z)^2 + (\delta x)^2} \sin\theta = 0 \tag{4}$$

but

$$\begin{aligned} sin\theta &= \frac{\delta z}{\sqrt{(\delta z)^2 + (\delta x)^2}} \\ cos\theta &= \frac{\delta x}{\sqrt{(\delta z)^2 + (\delta x)^2}} \end{aligned} \tag{5}$$

so that equation 4 becomes

$$F_x = (p_1 - p_3)b\delta z = 0 \quad p_1 - p_3 = 0 \tag{6}$$

Summing the forces in the z direction is similar, except that we now have to include the gravitational force acting on the mass of fluid in the prism:

$$F_z = p_2 b\delta x - p_3 b\sqrt{(\delta z)^2 + (\delta x)^2}\cos\theta - \frac{1}{2}\rho g b\delta z\delta x = 0 \tag{7}$$

Using equation 5 to eliminate θ and dividing through by $b\delta x$

$$F_z = (p_2 - p_3) - \frac{1}{2}\rho g\delta z = 0 \tag{8}$$

For finite values of δz $p_3 \neq p_2$, but in the limit as $\delta z \to 0$ the two must become equal. In physical terms, this means that as the fluid element shrinks, its volume decreases at a faster rate than its surface area, so that the weight force becomes negligible compared to the pressure force. Thus, in the limit, the pressures on the opposite faces of the element must become equal.

Equation 6 showed that hydrostatic pressure does not vary in a direction perpendicular to the gavity, since the equality of p_1 and p_3 holds for any value of δz. However, hydrostatic pressure can vary in the z direction. To see how it varies, consider the total force acting on a rectangular column of water extending from the surface to a point $z = -h$. As shown in figure 4.

$$F_z = -p_a\delta y\delta x + p(z)\delta y\delta x + \rho g z\delta y\delta x = 0 \tag{9}$$

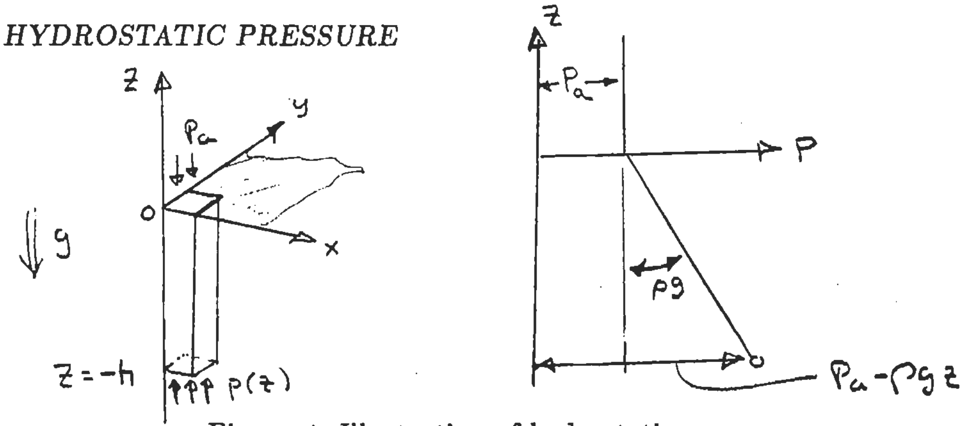


Figure 4: Illustration of hydrostatic pressure

so that

$$p(z) = p_a - \rho g z \qquad \frac{dp}{dz} = -\rho g \tag{10}$$

Hydrostatic pressure therefore starts out at atmospheric pressure on the surface, and increases linearly with depth, with a pressure gradient equal to the negative of the weight density of the fluid. The minus sign is simply because we chose to have the positive z direction point upward.

At the surface, the pressure is 14.7 pounds per square inch. At a submergence of 32 feet, the pressure in 59 deg F sea water, using English units is

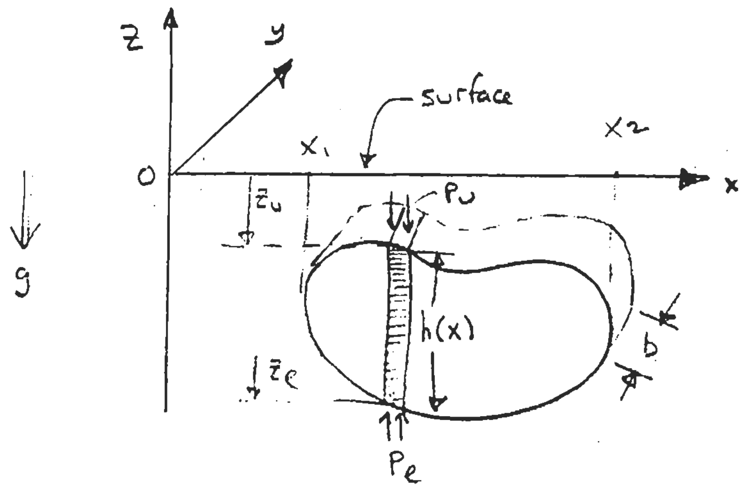
$$p(-32) = 14.7 - 1.9905 \times 32.174 \times (-32)/144 = 28.9 \quad pounds/in^2 \tag{11}$$

which is about double atmospheric pressure. Hence, as every SCUBA diver knows, the pressure increases by approximately one atmosphere for each 32 feet of submergence. At a depth of 20,000 feet, the pressure is 8,909 $pounds/in^2$ or 606 atmospheres.

NOTE. There is no section 4 in this edition -

5 ARCHIMEDES PRINCIPLE

Knowing how hydrostatic pressure varies with depth, we can now derive Archimedes principle, which states that a submerged body is buoyed up by a force equal to the weight of fluid which it displaces. Consider a prismatic body of arbitrary form, and thickness b as shown in figure. The force in the z direction may be found by integrating the pressure around the body,



$$
\begin{aligned}
F_z &= b\int_{x_1}^{x_2} (p_l(x) - p_u(x))dx \\
&= \rho g b \int_{x_1}^{x_2} h(x)dx \\
&= \rho g \nabla
\end{aligned}
\qquad (12)
$$

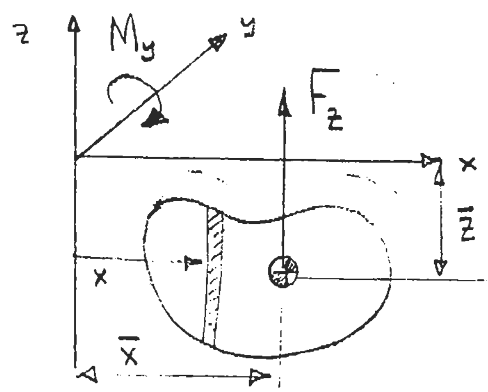
where ∇ is the volume of the submerged body and $h(x) = z_u - z_l$. The law also applies to floating bodies (whose volume protrudes through the free surface), except that the integral is limited to the submerged part of the volume.

The same analysis could be applied to a general three-dimensional body, but since we know this is the right answer, why not stop here?

We can also compute the moment of the buoyant force about the y axis,

$$\begin{aligned} M_y &= -b \int_{x_1}^{x_2} (p_l(x) - p_u(x)) x dx \\ &= -\rho g b \int_{x_1}^{x_2} h(x) x dx \\ &= -\bar{x}\nabla \end{aligned} \tag{13}$$

where $\bar{x}$ is the x coordinate of the resultant buoyant force. The negative sign is required in order to conform to a right-handed convention for moments.



Suppose, for the time being, that the orientation of our coordinate system is changed so that gravity points in the $+x$ direction. Then, the z coordinate of the resultant buoyant force will be

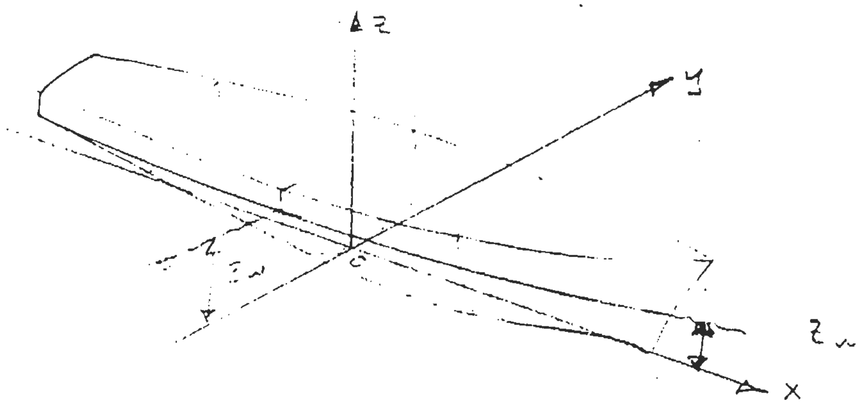
$$\begin{aligned} M_y &= -b \int_{z_1}^{z_2} (p_l(x) - p_u(x)) z dz \\ &= -\rho g b \int_{z_1}^{z_2} h(z) z dz \\ &= -\bar{z}\nabla \end{aligned} \tag{14}$$

Thus, the buoyant force always acts through the centroid of the volume. The centroid of a submerged volume and the *center of buoyancy* are therefore identical.

All these results can be deduced in a much more simple minded way by considering that the pressure distribution around body immersed in a fluid is the same as it would be if the body were just more of the same fluid. The resultant of the pressure integrated around the body surface must therefore yield a force which exactly equals the weight of the fluid, and acts through the center of mass of the fluid.

6 GEOMETRICAL PROPERTIES OF A HULL

Shown below is a sketch of a ship hull. The xz plane is the longitudinal plane of symmetry, and the z axis again points upward. However, the origin of the coordinates is not on the water surface, but is located at some convenient reference point on the hull (most generally at the bottom). The water surface, which we will allow to vary, is at $z = z_w$.

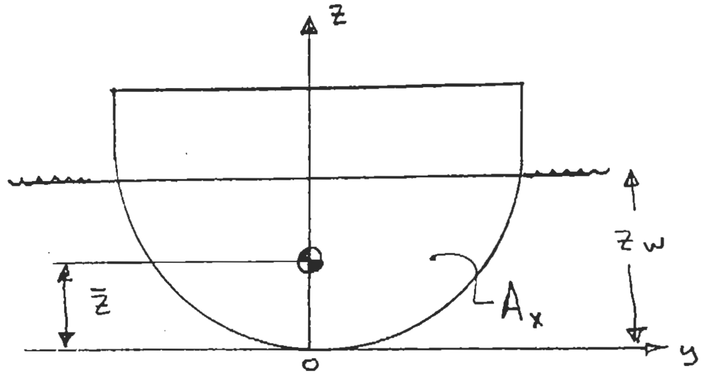


If we intersect the hull with a plane $x = const$, we get a *section* with area

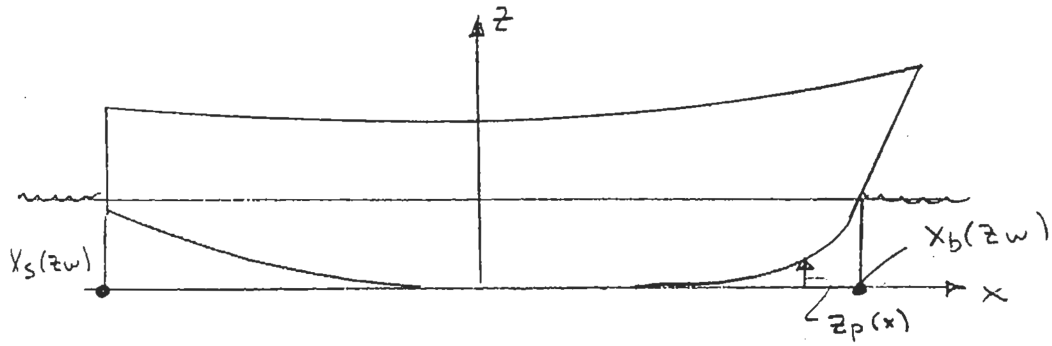
$$A_z(x, z_w) = 2 \int_{z_p(x)}^{z_w} y(x, z) dz \tag{15}$$

and centroid

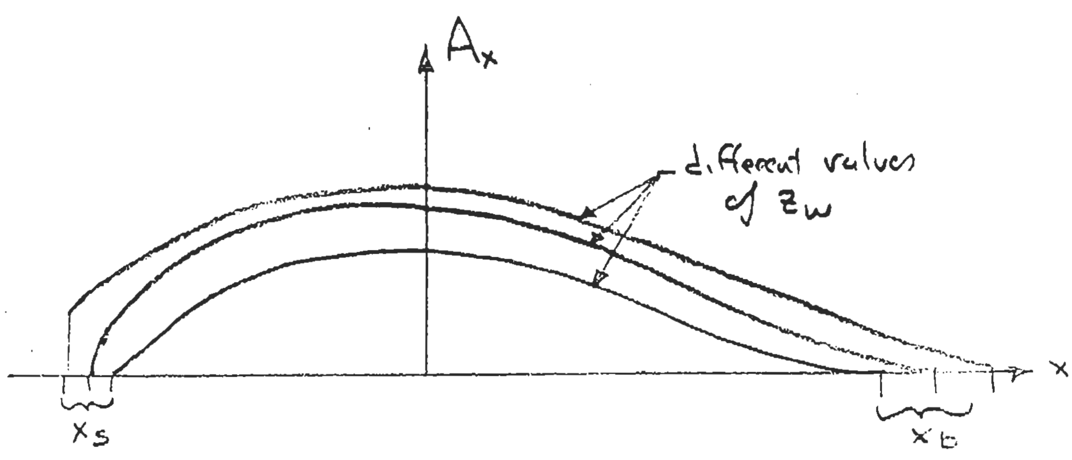
$$\bar{z}(x, z_w) = \frac{2 \int_{z_p(x)}^{z_w} z y(x, z) dz}{A_x(x, z_w)}$$
$$\bar{y}(x, x_w) = 0 \qquad (16)$$



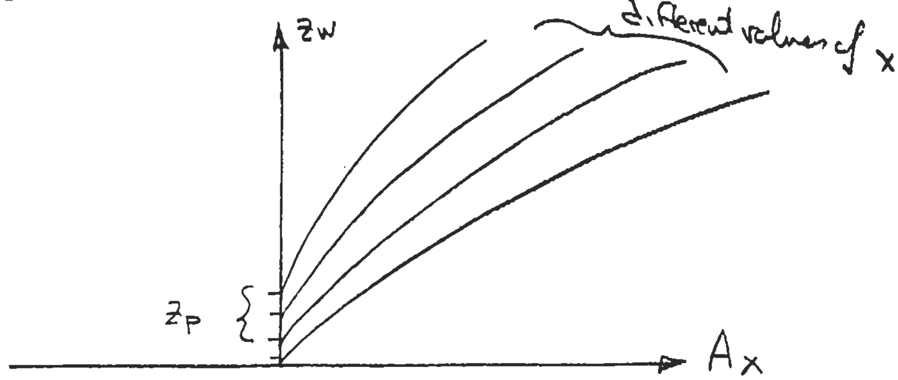
The *profile height* $z_p(x)$ is illustrated below



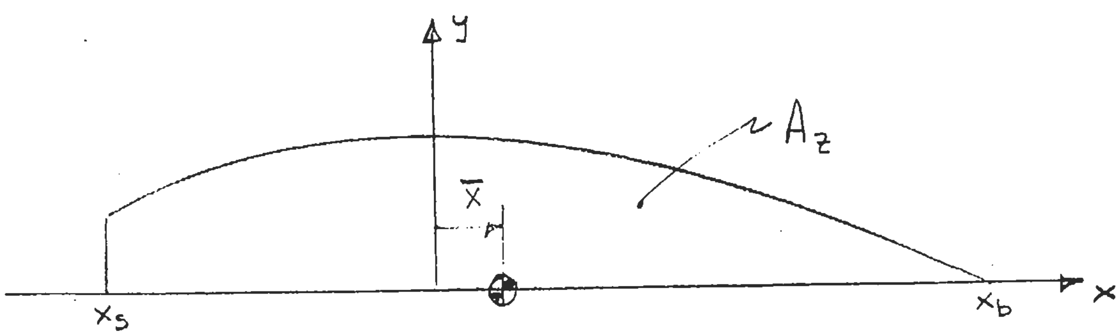
Plotted below are a set of sectional area curves for different values of z_w.



Another way to plot this same information is to plot $A(x)$ vs z_w for different values of x. These carry the name *Bonjean curves*, and used to be useful in graphically interpolating sectional areas for different waterlines.



If we slice the hull at constant values of z we obtain a waterline as shown below,



with waterplane area $A_z(z)$

$$A_z(x,z) = 2 \int_{x_s(z)}^{x_b(z)} y(x,z)dx \qquad (17)$$

with centroid

$$\bar{x}(z) = \frac{2 \int_{x_s(z)}^{x_b(z)} xy(x,z)dx}{A_z(z)}$$

$$\bar{y}(z) = 0 \tag{18}$$

We can now obtain the total volume of the hull, ∇, two different ways:

$$\nabla(z_w) = \int_{x_s(z_w)}^{x_b(z_w)} A_x(x, z_w)dx = \int_{z_p(min)}^{z_w} A_z(z)dz \tag{19}$$

where $z_p(min)$ is the lowest point on the profile.

Similarly, the x coordinate of the centroid of the immersed volume, which is commonly called the ***longitudinal center of buoyancy (LCB)***, we will call x_B,

$$x_B(z_w) = \frac{\int_{x_s(z_w)}^{x_b(z_w)} xA_x(x, z_w)dx}{\nabla} = \frac{\int_{z_p(min)}^{z_w} \bar{x}(z)A_z(z)dz}{\nabla} \tag{20}$$

The z coordinate of the centroid of immersed volume is called the *vertical (position) of the center of buoyancy (VCB)*, we will call z_B,

$$z_B(z_w) = \frac{\int_{x_s(z_w)}^{x_b(z_w)} \bar{z}(x, z_w)A_x(x, z_w)dx}{\nabla} = \frac{\int_{z_p(min)}^{z_w} zA_z(z)dz}{\nabla} \tag{21}$$

Some of these geometric quantities are useful when expressed in non-dimensional form. Define $\mathcal{L}$ $\mathcal{B}$ $\mathcal{T}$ as the length, beam and draft of the hull. Then the *block coefficient*, C_B is defined as the ratio of the actual volume of the hull to that of a block of dimensions $\mathcal{LBT}$,

$$C_B = \frac{\nabla}{\mathcal{LBT}} \tag{22}$$

A supertanker might have a block coefficient of 0.9 while a sailboat might have a value of 0.5.

A measure of how "pointed" the ends are is the *prismatic coefficient*,

$$C_P = \frac{\nabla}{A_z(x=0, z_w)\mathcal{L}} \tag{23}$$

which is the ratio of the actual volume to that of a prism with cross section equal to the sectional area at $x = 0$.

The *waterplane coefficient* is the ratio of the area of the waterplane to that of a rectangle,

$$C_W = \frac{A_z(z_w)}{\mathcal{L}\mathcal{B}} \tag{24}$$

and the *midships coefficient* is the ratio of the sectional area at $x = 0$ to a rectangle[1],

$$C_M = \frac{A_z(x=0, z_w)}{\mathcal{B}\mathcal{T}} \tag{25}$$

Carrying out the integrations formulated in this section is practical if the hull is smooth. However, if abrupt discontinuities in the geometry occur, one must be careful. Examples are the fins shown in the photograph of the submarine below, or the sailboat keel shown earlier in the course notes. In these cases, it is better to consider these as separate appendages. Their volumes and centers can be computed separately and combined with those of the main hull.

[1]The position of amidships is somewhat arbitrary, but we won't worry about that here

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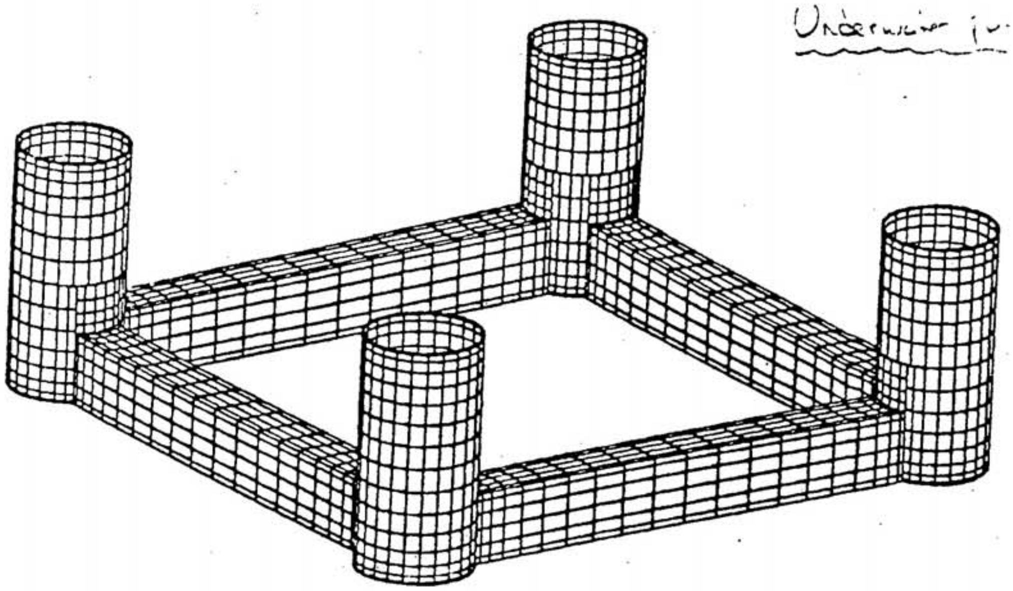
7 THINGS THAT AREN'T SHIPS

Hydrostatic properties of non-streamlined underwater vehicles, buoys, or floating offshore platforms are important. However, the traditional ship lines drawing approach to defining their volume properties is not necessarily appropriate. For example, slicing the offshore platform shown below into stations would be very inefficient. One approach in this case, is to ignore Archimedes, and to calculate the forces and moments by direct integration of the pressure over the surface. This can be done by dividing the surface into a large number of quadrilateral panels as shown in the plot below. This type of panel approximation is becoming widely used for hydrodynamic and structural calculations for offshore platforms.

Underwater [illegible]



Here's what one looks like above water. (not the same one)

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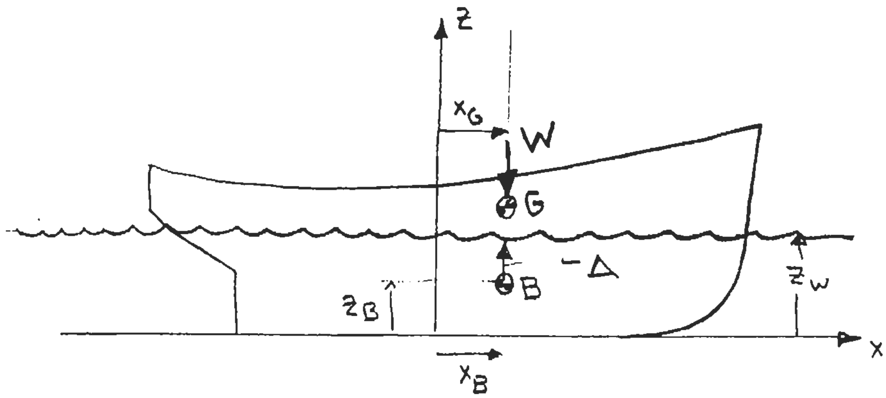
7 EQUILIBRIUM OF A FLOATING BODY

Supose an object is floating with a draft of z_w in the absence of any external forces, as shown below



The submerged volume is ∇, with centroid at x_B, z_B. The buoyant force is

$$F_z = \rho g \nabla = \Delta \tag{26}$$

where Δ is the symbol customarily given for the weight of the displaced volume of fluid, or just *displacement.* Since the net force in the z direction must be zero (or else it would accelerate up or down), the displacement must be exactly equal to the weight, W, of the body. Similarly, for equilibrium of moments about the y axis, the longitudinal position of the center of gravity, x_G must be equal to the longitudinal position of the center of buoyancy, x_B. The vertical positions of the centers of gravity and center of buoyancy, z_G and z_B, need not coincide. Intuitively, we would expect that if the center of gravity is too high, the floating body will capsize. However, that is a matter of stability, which we will be looking at soon. For now, all we can say is that the body will be in equilibrium for any vertical position of the center of gravity, but that a condition of *stable* equilibrium may not exist.

Getting to practical matters, if the weight of the object is given in English (long) tons, the weight density ρg must be expressed in tons per cubic foot, and the volume of displacement ∇ in cubic feet. For sea water at a termperature of 59 degrees Farenheit, we see from the table in Section 1 that $\rho = 1.9905/lbs - sec^2/ft^4$, and for a standard gravitational acceleration of $32.174/ft/sec^2$ so that $\rho g = 1.9905 \times 32.174/2240 = 0.0286\ tons/ft^3$. The reciprocal of this constant is 34.98 ft^3/ton. This is generally rounded to 35 ft^3/ton, and this serves to define standard sea water in the English system.

In the SI system, for sea water, $\rho = 1025\ kg/m^3$, and $g = 9.80665\ m/sec^2$, so that $\rho g = 0.01005\ MN/m^3$. This value is customarily rounded to 0.01 MN/m^3 which is certainly not too hard to remember!

All right, here is an example. A whale watching vessel has a length of 100 feet, a beam of 20 feet and a draft of 8 feet. It has a block coefficient of $C_B = 0.52$. What does it weigh, in tons?

$$\begin{aligned} \nabla &= C_B \times \mathcal{LBT} \\ &= 0.52 \times 100 \times 20 \times 8 = 8,320\ ft^3 \\ \Delta &\doteq \rho g \nabla = 8,320/35 = 237.7\ tons \end{aligned} \tag{27}$$

In metric units, the length is $30.48/m$, the beam is $6.096/m$ and the draft is 2.4384 m (down to the last tenth of a millimeter).

$$\begin{aligned} \nabla &= C_B \times \mathcal{LBT} \\ &= 0.52 \times 30.48 \times 6.096 \times 2.4384 = 235.60\ m^3 \\ \Delta &= \rho g \nabla = 0.01 \times 235.60 = 2.356\ MN \end{aligned} \tag{28}$$

Since 1 $MN = 100.36\ tons$, converting the metric weight back to English tons gives $2.356 \times 100.36 = 236.4\ tons$. This is slightly different from the value of 237.7 that we got before, due to the use of rounded nominal values of sea water weight density.

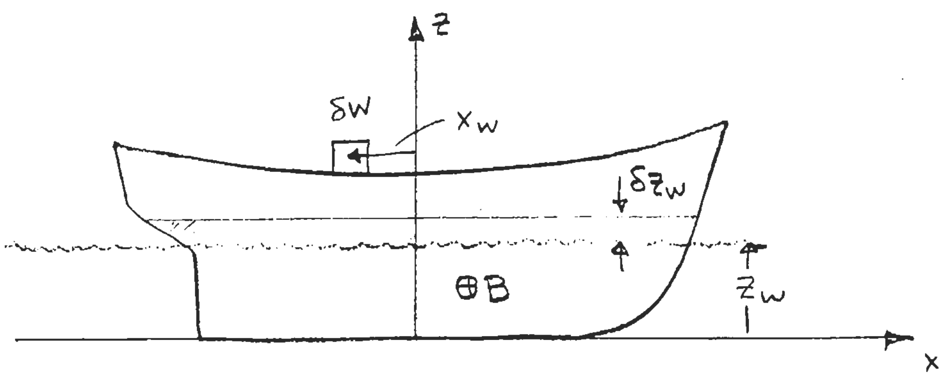
8 RESPONSE TO A SMALL WEIGHT ADDITION

8.1 Sinkage

Now suppose that we add an infinitesimal weight δw to the floating body. The body will sink down some small amount δz_w, and may also rotate about the y axis. The latter is called the ***trim angle*** θ. Presumably if the weight is added near the bow, that end will sink down more than the stern, and vice versa. However, for the time being, let us assume that the weight is added at just the right x location so that the trim angle is zero.

As shown in the sketch, for vertical equilibrium, the displacement must increase by δw. Thus, the increment in submerged volume must be such that $\rho g \delta \nabla = \delta w$. But,

$$\begin{aligned} \delta \nabla &\approx A_z(z_w)\delta z_w \\ \frac{\delta w}{\delta z_w} &\approx \rho g A_z(z_w) \end{aligned} \tag{29}$$



In the limit of vanishing added weight, this ratio, which is the ***weight per unit immersion*** becomes exact. In practice, this quantity is reasonably accurate for many purposes. In English units, the unit immersion is generally taken to be one inch. Using standard sea water density, the weight per unit immersion then becomes

$$\delta w = \rho g A_z(z_w) \times \frac{1}{12} = A_z(z_w)/420 \tag{30}$$

When expressed in these units, this quantity is known as the *tons per inch immersion* or TPI. The same quantity in metric units is the *meganewtons per meter immersion*

$$\delta w = \rho g A_z(z_w) \times 1 = 0.01A(z_w) \tag{31}$$

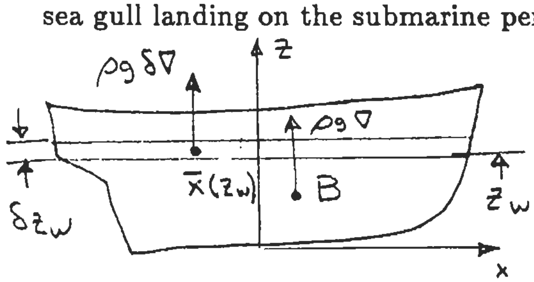
where the waterplane area, in this case, is expressed in square meters.

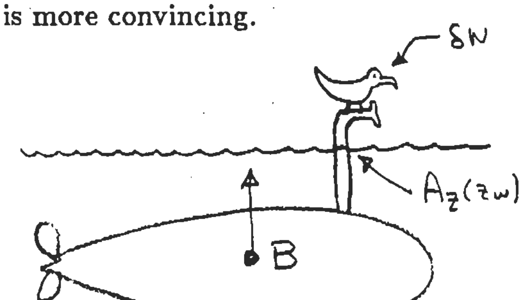
Back to the whale watching ship— Twelve Norwegian bachelor farmers, on a trip east from Minnesota join the group of passengers on board. How much does the draft increase? They each weigh 220 pounds, and the ship has a waterplane coefficient $C_w = 0.70$ at a draft of 8 feet.

$$\begin{aligned} 12\ farmers\ @\ 220\ lbs &= 1.179\ tons \\ A_z(z_w) &= 0.70 \times \mathcal{L}\mathcal{B} = 1400\ ft^2 \\ TPI &= A_z/420 = 1400/420 = 3.33 \\ \delta z &= \delta w/TPI = 1.179/3.33 = 0.354\ inches \end{aligned} \tag{32}$$

8.2 Trim

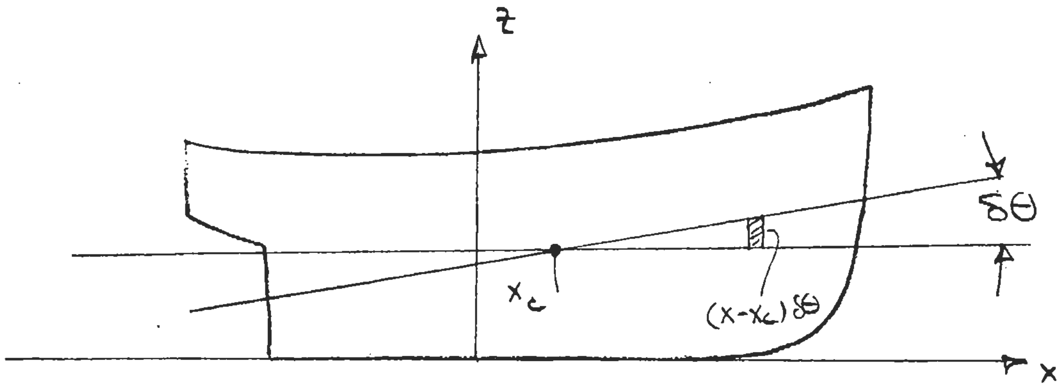
We assumed in the previous section that the weight was added at the x location which would result in no trim. Our next task is to find that point. A quick response would be that this is the longitudinal position of the center of buoyancy, x_B, but this generally not true. The additional buoyancy which supports the weight comes from a volume $A_z(z_w)\delta z$ whose longitudinal center is (approximately) at the centroid of the waterplane, $\bar{x}(z_w)$. Therefore, as shown in the sketch, moments about the y axis are balanced if the weight is added at this position. This might seem like a subtle difference for a ship, where these centers may not differ very much. Perhaps, the sea gull landing on the submarine periscope is more convincing.





The centroid of the waterplane is therefore a significant quantity, and is given the name *center of floatation*, LCF. Thus, $\bar{x}(z_w)$ and LCF mean the same thing.

We next consider the response of a floating body to an ininitesimal pure moment, δM_y about the y axis. Our first step is to determine the axis about which the object trims. Since we are applying a pure moment, there is no external force, and hence, the displacement must remain constant. Let us designate the position of the axis of rotation x_c, and compute the increment of volume $\delta\nabla$ associated with an increment in angle $\delta\theta$.



$$\begin{aligned}
\delta\nabla &= 2\int_{x_s}^{x_b}(x-x_c)\delta\theta y dx \\
&= 2\delta\theta\left[\int xydx - x_c\int ydx\right] = 0 \\
x_c &= \frac{2\int xydx}{A_z}
\end{aligned} \tag{33}$$

But this is the equation for the centroid of the waterplane. Hence $x_c = x_f$, so that a floating body trims about an axis through the center of floatation. Of course, this is only true for infinitesimal disturbances.

The hydrostatic moment caused by a small trim angle $\delta\theta$ is:

$$\delta M_y = -\int(x-x_f)xy\delta\theta dx$$

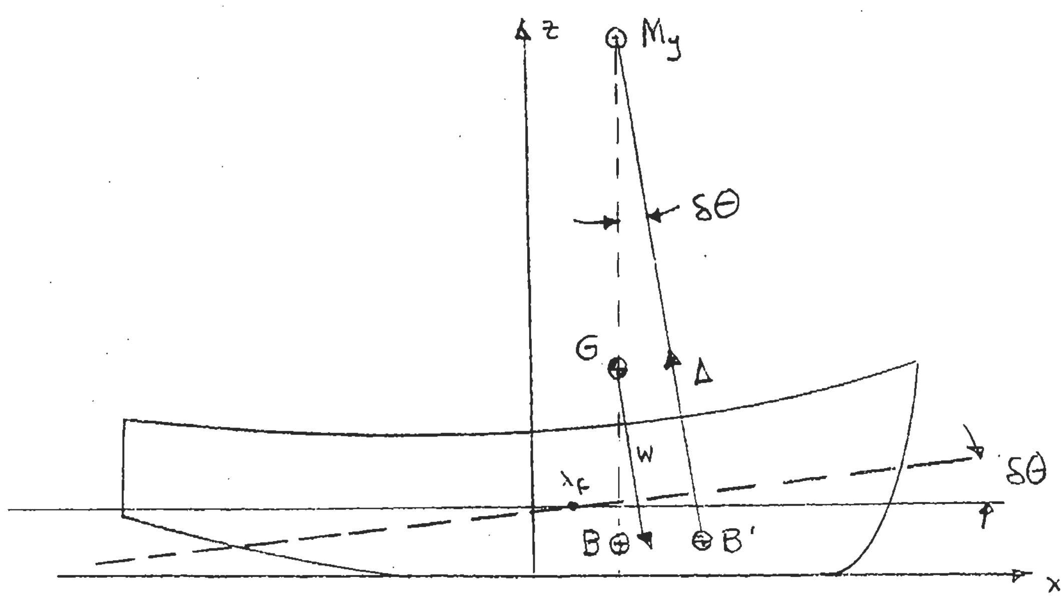
$$
\begin{aligned}
&= -2\rho g\delta\theta \int x^2 y dx + 2\rho g x_f \delta\theta \int xydx \\
x_f &= \frac{2\int xydx}{A_z} \\
I_y &= 2\int x^2 ydx \\
\delta M_y &= -\rho g \delta\theta \left[I_y - x_f^2 A_z\right] \\
\frac{\delta M_y}{\delta\theta} &= -\rho g I_y \qquad (34)
\end{aligned}
$$

We now have the simple result that the rate of change of hydrostatic moment with trim angle is proportional to the moment of inertia of the waterplane, taken about an axis through its centroid. This is labelled $\bar{I}_y$. It can be obtained from the moment of inertia about the y axis using the parallel axis theorem by subtracting $x_f^2 A_z$. This means that in addition to volumes and centers, we must add second moments (i.e. moments of inertia) to our library of geometric properties of floating bodies.

So far, we have only considered the hydrostatic moment. As shown in the figure below, as a floating body trims, the centers of buoyancy and gravity are no longer in line (with respect to the direction of gravity) so that an additional moment

$$\delta M_y = \Delta \overline{BG} \delta\theta \qquad (35)$$

exists which has a positive sign if G is above B and is therefore destabilizing.



The hydrostatic moment, which we computed from the moment of the immersed and emerged waterplane wedges, could also be described in terms of a shift of the center of buoyancy of the complete submerged volume from its original location x_B to a new location x'_B. We can therefore express the hydrostatic moment two different ways,

$$\delta M_y = -\Delta(x'_B - x_B) = -\rho g \bar{I}_y \delta\theta \tag{36}$$

Remembering that $\Delta = \rho g \nabla$, the longitudinal shift in the center of buoyancy is therefore

$$x'_B - x_B = \frac{\bar{I}_y}{\nabla}\delta\theta \tag{37}$$

This can be given a geometrical interpretation, as shown in the same figure. Under the initial condition of zero trim, the buoyant force acts vertically (in the positive z direction) through B. After an infinitesimal trim $\delta\theta$, the buoyant force acts through B', but in a direction inclined at an angle $\delta\theta$ from the z axis (which is fixed on the body, not in space). The new buoyant force vector intersects the original vertical line through B at a point which we will designate the ***longitudinal metacenter***, M_y. The distance between B and M_y is the ***longitudinal metacentric radius***, $\overline{BM}_y$.

$$\overline{BM}_y = (x'_B - x_B)/\delta\theta = \frac{I_y}{\nabla} \tag{38}$$

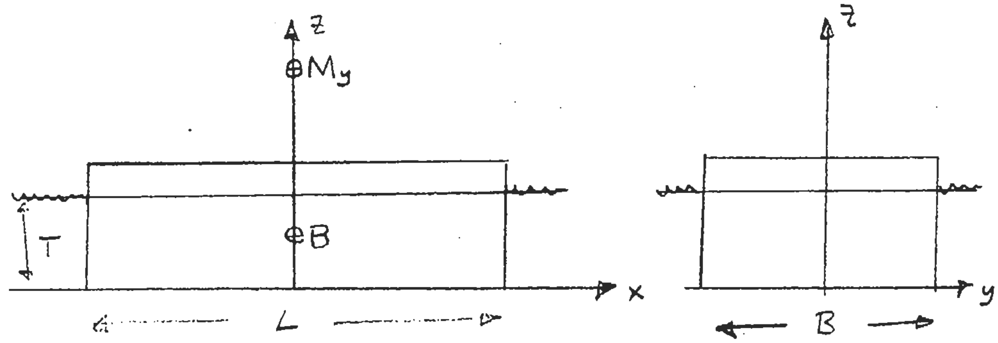
The total moment due to trim, including both the hydrostatic component and the moment due to the vertical distance between B and G is

$$(\delta M_y)_{total} = \Delta[\overline{BG} - \overline{BM}_y]\delta\theta = -\Delta\overline{GM}_y\delta\theta \tag{39}$$

The resultant moment has a negative sign (which is stabilizing) if the metacenter is above the center of gravity. The distance between the center of gravity and the metacenter, $\overline{GM}_y$ is called the ***metacentric height.*** If a floating body has a positive metacentric height, it is stable with respect to small angular disturbances. That is, a small angular displacement in trim, $\delta\theta$ will result in a net moment about the y axis which is in a direction to restore the body to its initial orientation. Note that the center of gravity does not need to be below the center of buoyancy for the body to be stable.

The metacenter is analogous, in a way, to the center of gravity. No matter how an object is oriented, the force of gravity acts through it's center of gravity. The force of buoyancy, of course, acts though the center of buoyancy. However, as a floating body is displaced, the center of buoyancy moves. In the limit of small angular displacements, however, the buoyant force acts through a *fixed* point fixed on the body, which we call the metacenter.

It is time to look at a couple of specific examples. Suppose that we have a box shaped floating body with dimensions $\mathcal{L}, \mathcal{B}, \mathcal{T}$, as shown below.



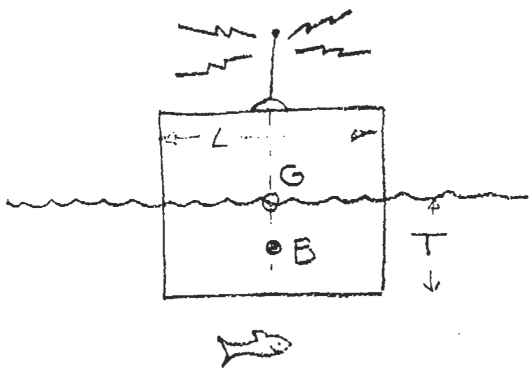
Since the origin of our coordinate system is centered with respect to the length, $x_f = 0$. Knowing the formula for the moment of inertia of a rectangle (or looking it up, or deriving it), we can find the metacentric radius without much difficulty,

$$
\begin{aligned}
\bar{I}_y &= \frac{1}{12}\mathcal{L}^3\mathcal{B} \\
\nabla &= \mathcal{L}\mathcal{B}\mathcal{T} \\
\overline{BM}_y &= \frac{\bar{I}_y}{\nabla} = \frac{\mathcal{L}^2}{12\mathcal{T}} \\
\frac{\overline{BM}_y}{\mathcal{L}} &= \frac{1}{12}\frac{\mathcal{L}}{\mathcal{T}}
\end{aligned}
\tag{40}
$$

The last line is instructive, since it shows that the non-dimensional metacentric radius is simply proportional to the ratio of length to draft of the floating rectangular body. For typical ship proportions, the ratio of $\mathcal{L}/\mathcal{T}$ is 12 or greater. Therefore, the longitudinal metacentric radius for ships is of the order of their length. Since ships generally have a low center of gravity, the distance between the center of gravity and the center of buoyancy $\overline{BG}$ is small, say of the order of $\mathcal{T}$. This means that ships are always extremely stable with respect to small displacements in trim angle. We will see later that ships are much less stable with respect to angular displacements about the x axis (as anybody who has jumped into a canoe knows), so that the latter is generally critical with regards to safety.

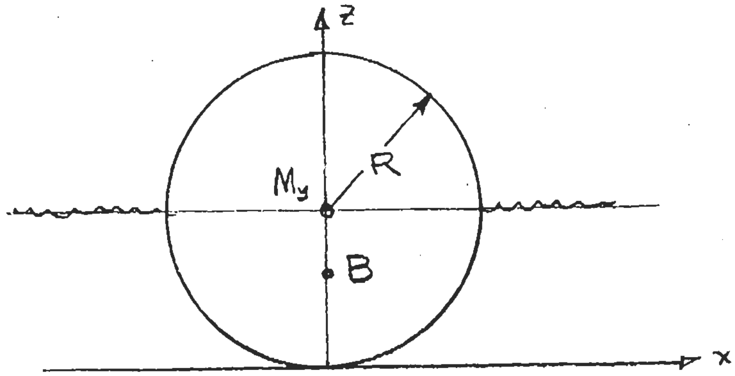
On the other hand, offshore drilling platforms might have lower values of $\mathcal{L}/\mathcal{T}$, and *much* higher values of the center of gravity. In that case, stability with respect to trim angle might be of much greater concern.

Buoys and floating instrument packages are another example. As illustrated below, an instrument detects the presence of red and yellow striped fish as they swim by, and transmits a radio signal each time to a satellite. The instrument is packaged in a square container and has a specific gravity of 0.5 relative to sea water, with a center of gravity in the middle of the box. Is it stable as shown?



Since the specific gravity is 0.5, it floats with half of the volume under water. Thus $\mathcal{L}/\mathcal{T} = 2$ and $\overline{BM_y}/\mathcal{L} = 1/6$ and $\overline{BM_y}/\mathcal{T} = 1/3$. Since the center of buoyancy is at half the draft, $\overline{BG} = \mathcal{T}/2$. Thus, we see that the metacenter is *below* the center of gravity (by $\mathcal{T}/6$) so that the device is unstable. Better tell the designer to put some lead ballast in the bottom of the package.

The second example is circular cylinder of radius R floating on its axis, as shown below



It is obvious in this case that the center of buoyancy will not move when the body is subjected to a trim angle displacement (small or not), and that the metacenter must therefore coincide with the axis. Let's calculate it anyway.

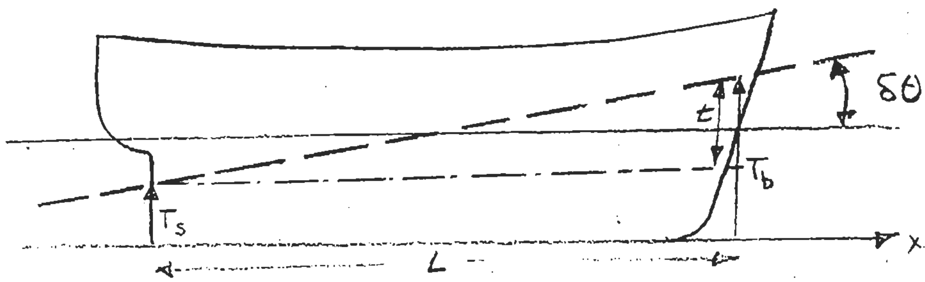
$$\begin{aligned} \bar{I}_y &= \frac{1}{12}(2R)^3 B = \frac{2}{3}R^3 B \\ \overline{BM}_y &= \bar{I}_y/\nabla = \frac{4R}{3\pi} \end{aligned} \tag{41}$$

but we also know that the centroid of a semi-circle is at a distance $(4R)/(3\pi)$ from its axis ($z_B = R(1-4/(3\pi))$), which confirms the fact that the metacenter is at the waterline. We now know that homogeneous logs of specific gravity 0.5 are *neutrally stable*, since their center of gravity and metacenter coincide. A semi-circular shape is therefore only stable if its center of gravity is below the waterline.

9.2 Moment per unit trim

We saw in the preceding section that for floating bodies that are long with respect to their draft, and have low centers of gravity, $\overline{BM}_y \gg \overline{BG}$. In this case, the height of the center of gravity is unimportant in determining the moment due to trim angle. We can therefore make the approximation that $\overline{GM}_y \approx \overline{BM}_y$. In this case the response to a trim disturbance becomes a geometric property of the body itself, and is independent of its weight distribution. One can therefore derive simple to apply expressions for the trim response of a floating body, in a manner similar to the *tons per inch immersion* derived previously.

To begin with, trim angle $\delta\theta$ is hard to deal with in the real world since small angles are hard to measure. A more convenient quantity is ***trim***, t, which is defined as the difference in draft at the bow and at the stern, $t = T_b - T_s$ as shown below.



Ships generally have draft marks painted at the bow and stern at regular intervals, so that the operator can simply read the two values and subtract them to determine the trim. For small angles, $\delta\theta \approx t/\mathcal{L}$.

In English units, one can define the ***moment to trim one inch***, MTI, as

$$\begin{aligned} MTI &\equiv \delta M_y \approx \Delta\overline{BM}_y\frac{1}{12\mathcal{L}} \\ &= \frac{\rho g\bar{I}_y}{12\mathcal{L}} = \frac{\bar{I}_y}{420\mathcal{L}} \end{aligned} \tag{42}$$

where the constant $420 = 35 \times 12$ contains the standard weight density for sea water.

It's time to get back to the Norwegian bachelor farmers on the whale watching ship. They all eagerly walk up to the bow to get a better look at the whales. (It is too cold up there for the rest of the passengers). How much does the ship trim? If the waterplane were a rectangle, its inertia would be

$$\bar{I}_y = \frac{1}{12} \times 100^3 \times 20 = 1,666,667\ ft^4 \tag{43}$$

However, since it is a more streamlined shape ($C_W = 0.70$), we will assume that the real moment of inertia is half of that value. The moment to trim one inch is therefore

$$MTI = (\frac{1}{2})\frac{1,666,667}{420 \times 100} = 19.84 \tag{44}$$

If the twelve farmers move half the length of the ship from the center of floatation, the trim moment is $12 \times 220 \times 50/2240 = 58.93\ ft - tons$. The trim is therefore $58.93/19.84 = 2.97\ inches$. The trim angle is $2.97/100 = 0.0297\ radians = 1.7\ degrees$.

In the metric system, the equivalent quantity is the *meganewtons per meter trim.* Using the nominal weight density of sea water, the moment to trim one meter is $0.01\bar{I}_y/\mathcal{L}$.

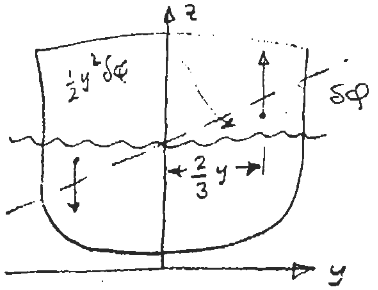
9.3 Heel

Rotation about the x axis is called *heel angle* or sometimes roll angle, although the latter implies a dynamic, rather than a static quantity. The symbol ϕ is used to denote this angle, and we will use $\delta\phi$ to denote an infinitesimally small displacement in heel.

By a simple substitution of coordinates, the hydrostatic moment due to heel can be obtained from our previous result for trim,

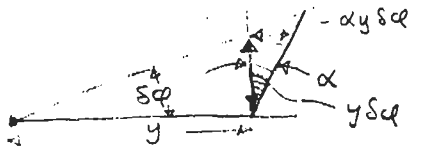
$$\frac{\delta M_x}{\delta\phi} = -\rho g \bar{I}_x \tag{45}$$

If we assume symmetry about the xz plane, the center of floatation is at $y = 0$ so we don't need the bar over the I_x. We will derive the formula anyway, since it will illustrate the computation of waterplane inertia in a different order.



$$\delta M_x = -2\rho g \int (\frac{2y}{3}\frac{y^2}{2}\delta\phi dx$$
$$\frac{\delta M_x}{\delta\phi} = \frac{2}{3}\rho g \int y^3 dx = \rho g I_x \tag{46}$$

In the above figure, the sides of the body were drawn vertically in the vicinity of the free surface, so that the approximation of the incremental buoyancy wedges by right triangles was reasonable. What if the body is narrow, and has extreme slope to the sides at the waterline? Sketched below is a waterline wedge showing the sides making an angle α with respect to the vertical. The additional wedge area on one side due to the sloping side is approximately



$$\frac{1}{2}\alpha y^2 \delta\phi^2 \tag{47}$$

and the moment of the wedge on one side is approximately

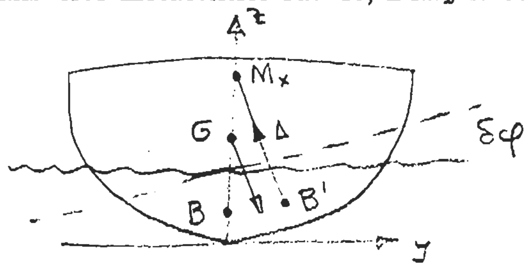
$$(y)(\frac{1}{2}\alpha y^2 \delta\phi^2) \sim \alpha y^3 \delta\phi^2 \tag{48}$$

Thus in the limit as $\delta\phi \rightarrow 0$, the contribution of the sloping sides to the wedge moment goes to zero. However, for finite values of the heel angle, the slope of the sides will affect the shape of the curve of hydrostatic moment versus heel angle, as we will see later.

Following the same argument as before, the buoyant force intersects a vertical line through the center of buoyancy at a point M_x, which is called the *transverse metacenter*, and the distance $\overline{BM}_x = I_x/\nabla$ is called the *transverse metacentric radius.* Finally, the combined hydrostatic and weight moment is

$$\delta M_x = -\Delta(\overline{BM}_x - \overline{BG}) \tag{49}$$

as illustrated below. Since ships generally have a ratio of beam to draft of around two to four, the transverse metacentric radius, $\overline{BM}_x$ is of the same order as the



distance between B and G. The height of the center of gravity is therefore critical in establishing transverse stability. The equivalent to *moment to trim one inch* is therefore not of any utility.

Back to the whale watchers. Our friends finally got cold up in the bow, walked back to $x = 0$, but then walked out to the rail on one side (possibly motivated by sea sickness). How much does the ship heel?

Again assuming a rectangular waterplane,

$$I_x = \frac{1}{12} \times 20^3 \times 100 = 66,667\ ft^4 \tag{50}$$

and assuming again that the real inertia is half that of a rectangle, we find that the transverse metacentric radius is

$$\overline{BM}x = \frac{I_x}{\nabla} = \frac{66,667}{8,320} = 8.01\ ft \tag{51}$$

Now we need to know where the center of gravity is in order to get the metacentric height. Remember that this is not a geometrical quantity of the hull that we can calculate, but comes from a detailed knowledge of the complete contents of the ship. Anyway, we are given the fact that $\overline{BG} = 6.0$

The heeling moment for 12 farmers moving out half of the beam is 11.786 $foot-tons$, so we have

$$\begin{aligned} \delta M_x &= \Delta \overline{GM}_x \delta\phi \\ \delta\phi &= \frac{11.786}{237.7 \times (8.01 - 6.00)} = 0.025\ radians \\ &= 1.41\ degrees \end{aligned} \tag{52}$$

11 HYDROSTATIC CURVES

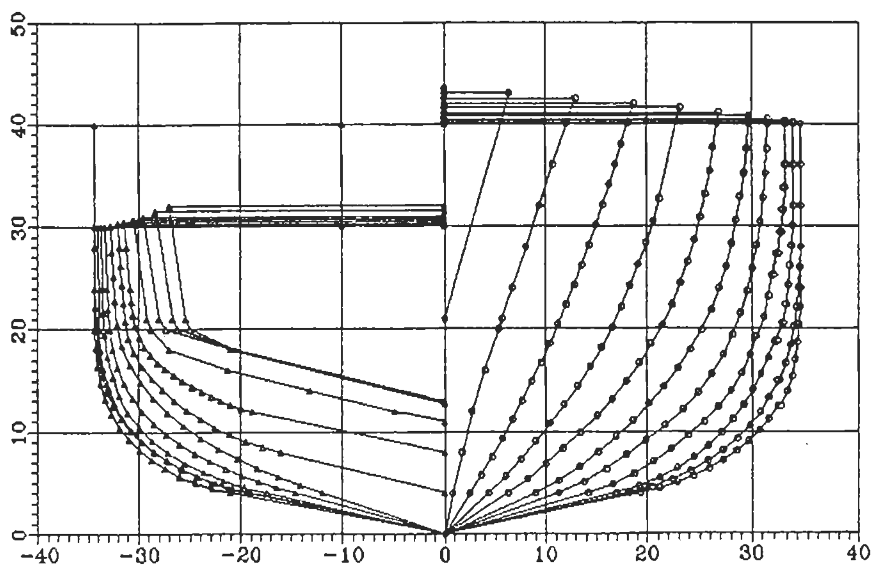
We saw from the preceding sections that we could calculate the response of a floating body to small disturbances given simple geometrical properties of its submerged volume and waterplane, and given the object's weight and center of gravity. Since these geometrical characteristics depend on the draft of the floating body, it is convenient to calculate them over a range of possible drafts, and then interpolate as necessary at the desired draft.

In the past, the interpolation process could be carried out most conveniently by having the hydrostatic data plotted to a large scale on graph paper. For some reason, it was accepted practice to plot everything on one large graph. Since different hydrostatic properties differ in units and magnitude, this meant that the scale to which each quantity was plotted had to be noted. Examples of this type of presentation can be found in traditional naval architectural texts.

In more recent years, tabulated computer generated hydrostatic data or on-board computers have replaced the old graphs. An example of tabulated data contained in an operating manual for a containership[1] is shown on the next page.

However, it is always nice to have graphs to see trends and to spot possible errors. In addition, regulatory agencies frequently require them in connection with safety reviews. Shown on the next few pages are small scale plots of the hydrostatic characteristics of an oceanographic ship. The sections of the ship used as input for these calculations are shown below.

[1] Macy,R.H.,"VESSEL DAMAGE CONTROL-SL-18 CONTAINER SHIPS", Sea Land Services, 1972



Hydrostatic Data

Keel Draft Feet	Keel Draft Inches	Displ. Tons	TPI Tons/Inch	MT1" Foot-Tons	KM Feet	LCB Feet	LCF Feet
32	0	34427	108.8	3741.6	39.00	325.88	314.03
32	1	34536	108.9	3752.8	39.01	325.84	313.92
32	2	34646	109.0	3764.0	39.02	325.80	313.81
32	3	34755	109.1	3775.2	39.03	325.76	313.70
32	4	34865	109.3	3786.5	39.03	325.72	313.59
32	5	34975	109.4	3797.8	39.04	325.68	313.49
32	6	35085	109.5	3809.2	39.05	325.64	313.38
32	7	35195	109.6	3820.6	39.06	325.60	313.27
32	8	35305	109.7	3832.1	39.07	325.56	313.16
32	9	35416	109.9	3843.6	39.08	325.52	313.06
32	10	35526	110.0	3855.2	39.09	325.48	312.95
32	11	35637	110.1	3866.8	39.10	325.44	312.84
33	0	35748	110.2	3878.4	39.11	325.40	312.74
33	1	35858	110.4	3890.1	39.12	325.36	312.63
33	2	35969	110.5	3901.8	39.13	325.32	312.53
33	3	36080	110.6	3913.5	39.14	325.28	312.42
33	4	36192	110.7	3925.2	39.15	325.23	312.32
33	5	36303	110.9	3937.0	39.16	325.19	312.21
33	6	36414	111.0	3948.8	39.17	325.15	312.11
33	7	36526	111.1	3960.6	39.18	325.11	312.01
33	8	36638	111.2	3972.5	39.20	325.07	311.91
33	9	36750	111.4	3984.3	39.21	325.03	311.81
33	10	36862	111.5	3996.2	39.22	324.99	311.71
33	11	36974	111.6	4008.1	39.23	324.95	311.61
34	0	37086	111.7	4020.0	39.25	324.90	311.51
34	1	37199	111.9	4031.9	39.26	324.86	311.41
34	2	37311	112.0	4043.8	39.27	324.82	311.31
34	3	37424	112.1	4055.8	39.29	324.78	311.22
34	4	37536	112.2	4067.7	39.30	324.74	311.12
34	5	37649	112.3	4079.6	39.32	324.70	311.03
34	6	37762	112.5	4091.6	39.33	324.65	310.93
34	7	37876	112.6	4103.5	39.34	324.61	310.84
34	8	37989	112.7	4115.4	39.36	324.57	310.75
34	9	38102	112.8	4127.4	39.37	324.53	310.66
34	10	38216	113.0	4139.3	39.39	324.49	310.57
34	11	38330	113.1	4151.2	39.40	324.45	310.48
35	0	38444	113.2	4163.1	39.42	324.40	310.39
35	1	38558	113.3	4175.0	39.44	324.36	310.31
35	2	38672	113.5	4186.9	39.45	324.32	310.22
35	3	38786	113.6	4198.7	39.47	324.28	310.14
35	4	38900	113.7	4210.5	39.48	324.24	310.06
35	5	39015	113.8	4222.4	39.50	324.20	309.98
35	6	39129	114.0	4234.1	39.52	324.15	309.90
35	7	39244	114.1	4245.9	39.53	324.11	309.82
35	8	39359	114.2	4257.6	39.55	324.07	309.75
35	9	39474	114.3	4269.4	39.57	324.03	309.67
35	10	39589	114.4	4281.0	39.59	323.99	309.60
35	11	39705	114.6	4292.7	39.60	323.95	309.53

Figure by MIT OCW.

RESEARCH VESSEL PRELIMINARY DESIGN DATE:11-26-89 TIME:15:20:51

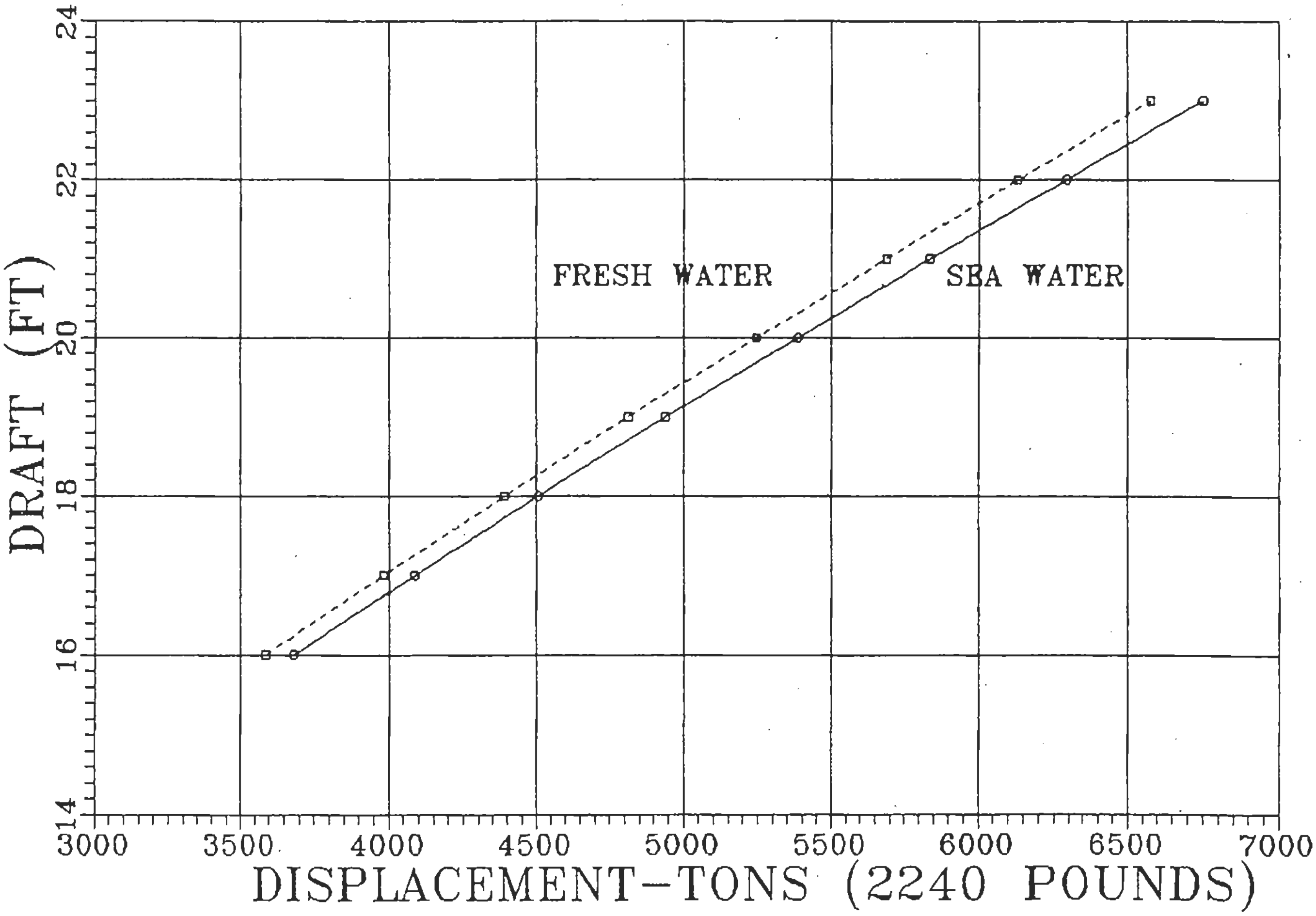
LONGITUDINAL INTEGRATION BY PARABOLIC RULE

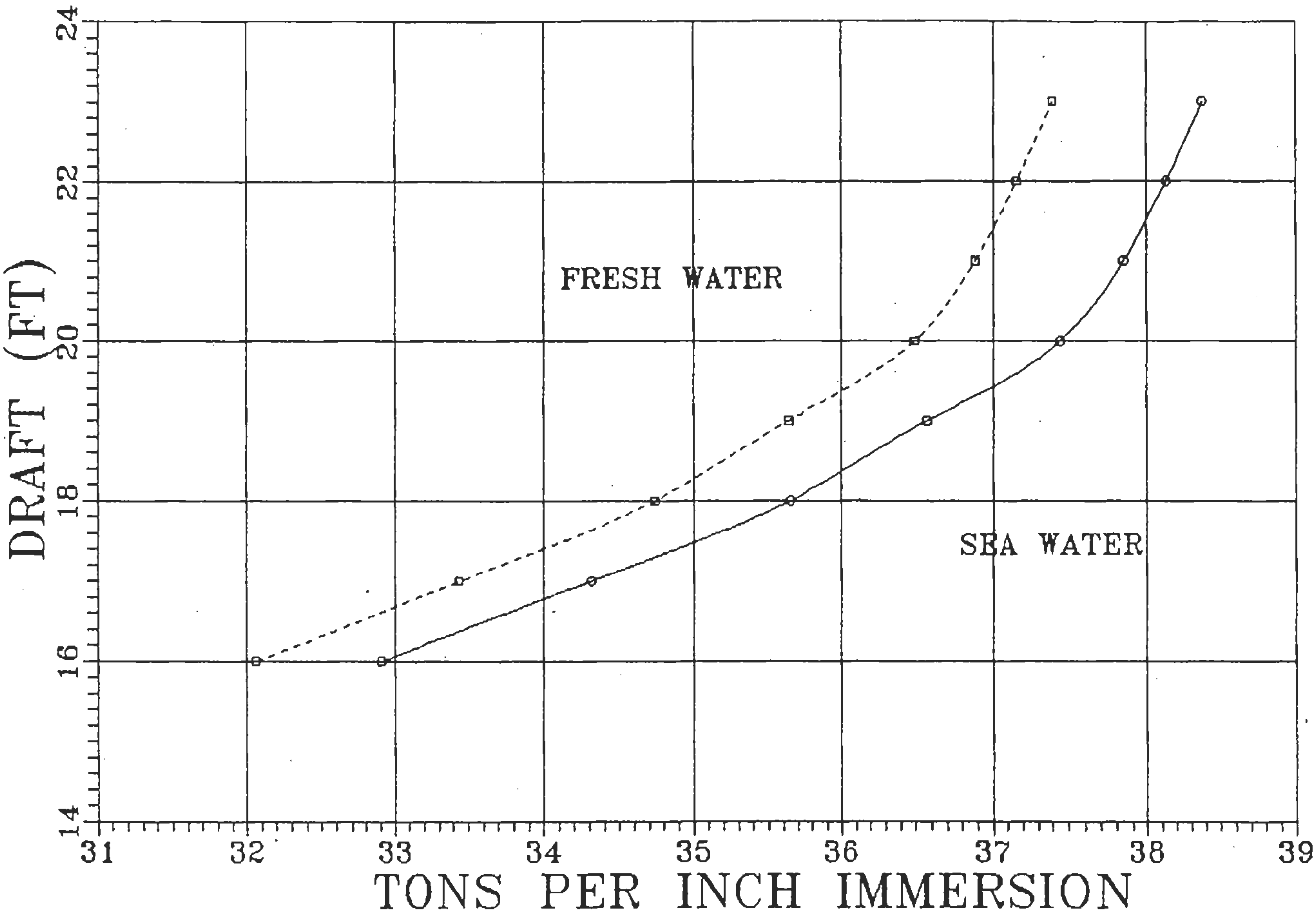
HYDROSTATIC PROPERTIES OF THE HULL

************** WL HEIGHT ABOVE BASE **************

FWD PERP	16.000	17.000	18.000	19.000	20.000	21.000	22.000	23.000
MID PERP	16.000	17.000	18.000	19.000	20.000	21.000	22.000	23.000
AFT PERP	16.000	17.000	18.000	19.000	20.000	21.000	22.000	23.000

DISP-TON-SW	3682.2	4086.1	4506.5	4940.3	5385.2	5837.9	6292.4	6751.5
DISP-TON-FW	3588.2	3981.8	4391.5	4814.2	5247.7	5688.8	6131.7	6579.1
TONS/IN-SW	32.90	34.31	35.65	36.57	37.45	37.85	38.13	38.38
TONS/IN-FW	32.06	33.44	34.74	35.64	36.49	36.89	37.16	37.40
FT-TON/IN S	508.98	560.09	608.91	642.84	675.81	689.89	700.32	710.24
FT-TON/IN F	495.99	545.79	593.37	626.42	658.56	672.27	682.44	692.11
CB ABV BASE	10.15	10.78	11.40	12.03	12.64	13.25	13.85	14.44
TRANS KM	39.15	38.71	38.54	38.08	37.81	37.01	36.21	35.53
LONG KM	497.82	494.36	488.10	471.09	455.39	430.17	406.50	385.58
LONG C.B.	2.58	1.24	-.12	-1.42	-2.62	-3.68	-4.62	-5.39
LONG C.F.	-9.97	-12.40	-14.61	-15.70	-16.72	-16.51	-16.07	-15.66
WETTED SURF	16635.1	17556.4	18464.5	19233.2	19997.3	20614.8	21266.4	21871.1
BLOCK COEFF	.403	.419	.435	.452	.467	.482	.496	.509
LONG PRISM	.539	.552	.565	.578	.590	.602	.612	.622
MIDSHIPS	.748	.760	.771	.782	.792	.802	.811	.819
VERT PRISM	.583	.584	.585	.592	.599	.612	.625	.637
WATERPLANE	.692	.718	.744	.762	.780	.788	.794	.799




DRAFT (FT)
TONS PER INCH IMMERSION
FRESH WATER
SEA WATER
24
22
20
18
16
14
31
32
33
34
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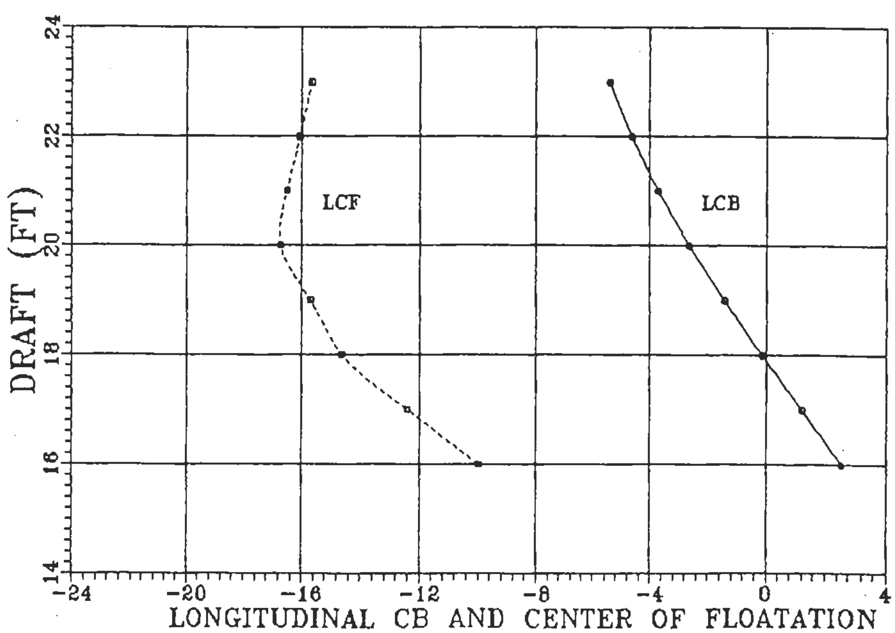
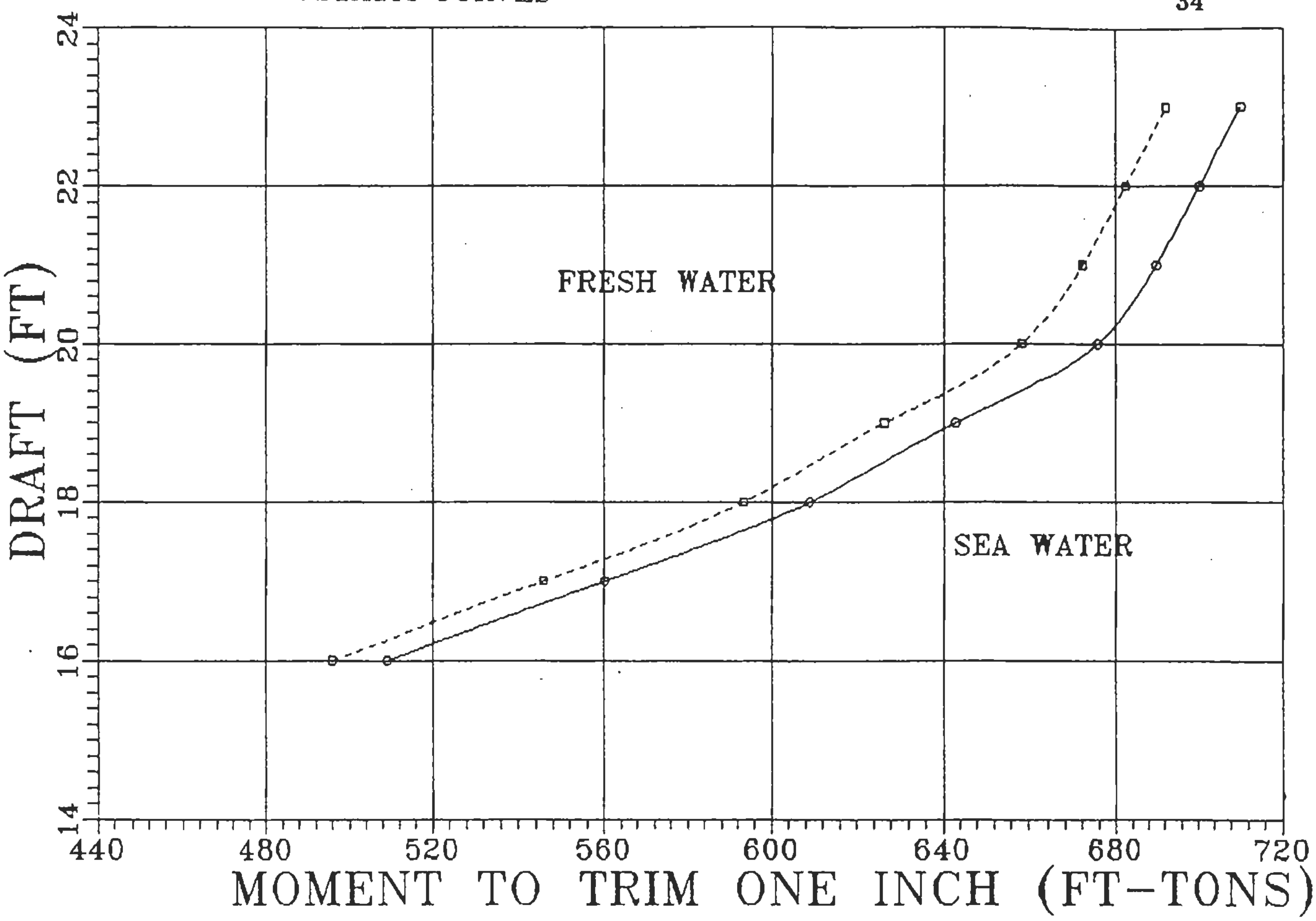
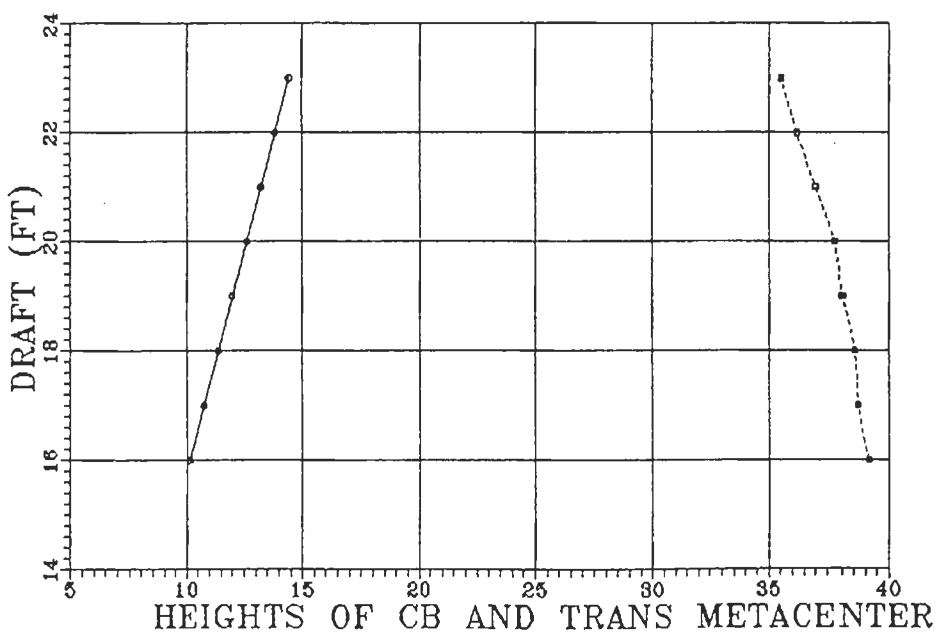
36
37
38
39


DRAFT (FT)
LONGITUDINAL CB AND CENTER OF FLOATATION
LCF
LCB
24
22
20
18
16
14
-24
-20
-16
-12
-8
-4
0
4


DRAFT (FT)
MOMENT TO TRIM ONE INCH (FT-TONS)
FRESH WATER
SEA WATER
14
16
18
20
22
24
440
480
520
560
600
640
680
720


DRAFT (FT)
HEIGHTS OF CB AND TRANS METACENTER
14
16
18
20
22
24
5
10
15
20
25
30
35
40

12 LARGE WEIGHT CHANGES

We can use the hydrostatic curves to solve problems involving weight changes that are too large for the small weight approximations to be valid. Here is an example. Suppose the oceanographic ship has just been built and is floating empty at a level draft of 16 feet. A total of 2000 tons of equipment is added at longitudinal position $x_w = -5.0\ ft$. Find the resulting draft and trim.

The small weight approximation would be to find the tons per inch immersion at the 16 ft draft (32.9), and get

$$\begin{aligned} \delta z &= 2000/32.9 = 60.79\ inches\ = 5.06\ ft \\ z_w &\rightarrow z_w + \delta z = 16 + 5.06 = 21.06 \end{aligned} \tag{54}$$

Instead, we know that the displacement at the original 16 ft draft is 3682 tons, so that the new displacement is $\Delta = 3682 + 2000 = 5682$. Entering the graph of displacement versus draft at this value, we find that the new draft is 20.66 $ft.$[2]. Hence, the small weight approximation overestimated the draft by 0.44 ft. This is to be expected, since the waterplane area (and hence the tons per inch) increases with draft.

Now we have to deal with the trim. Using the small weight approximation, the center of floatation at the initial draft is $x_f = -9.97\ ft$ and the moment to trim one inch is 508.98. The approximate trim is therefore

$$t = -2000 \times (-9.97 + 5.00)/508.98 = 19.53\ inches \tag{55}$$

A more accurate way to do it is first to calculate the new longitudinal position of the center of buoyancy,

[2] I must admit that I couldn't read the graph to that accuracy, but I have the program and you don't!

$$\begin{aligned}(3682+2000)\times x'_G &= 3682\times x_G + 2000\times x_W\\ x'_G &= \frac{3682\times 2.58 + 2000\times -5}{5682} = -.09\ ft\end{aligned} \tag{56}$$

and then calculate the unbalanced moment as the product of the new total displacement and the distance between the new center of buoyancy and the new center of gravity,

$$\begin{aligned}\delta M_y &= -\Delta(x'_B - x'_G)\\ &= 5682\times(3.35-0.09) = 18,523\ ft-tons\end{aligned} \tag{57}$$

So far this is exact. We must now make the assumption that the trim is small, but we will use the moment to trim one inch at the new draft. Reading the curves, this is 684.81. The trim is then

$$t = 18,523/684.81 = 27.0\ inches \tag{58}$$

compared with 19.5 inches obtained before. One reason why the trim results are so different is that the position of the center of floatation changes rapidly with draft for this particular hull shape.

13 STABILITY AT LARGE ANGLES

We saw that ships are generally much more stable with respect to trim than with respect to heel. This is a direct consequence of their large ratio of length to breadth. As a result, trim angles seldom get to be very large, and the small trim approximation developed earlier is good enough in most cases. On the other hand, heel angles can become very large, and in some cases we need to know a ship's stability characteristics all the way to ±180 degrees. The small angle approximation is therefore useless for this purpose.

Offshore platforms, buoys and other *non-ship like* objects may have length to beam ratios close to one. In this case, large displacements in either heel or trim (or combinations) are possible.

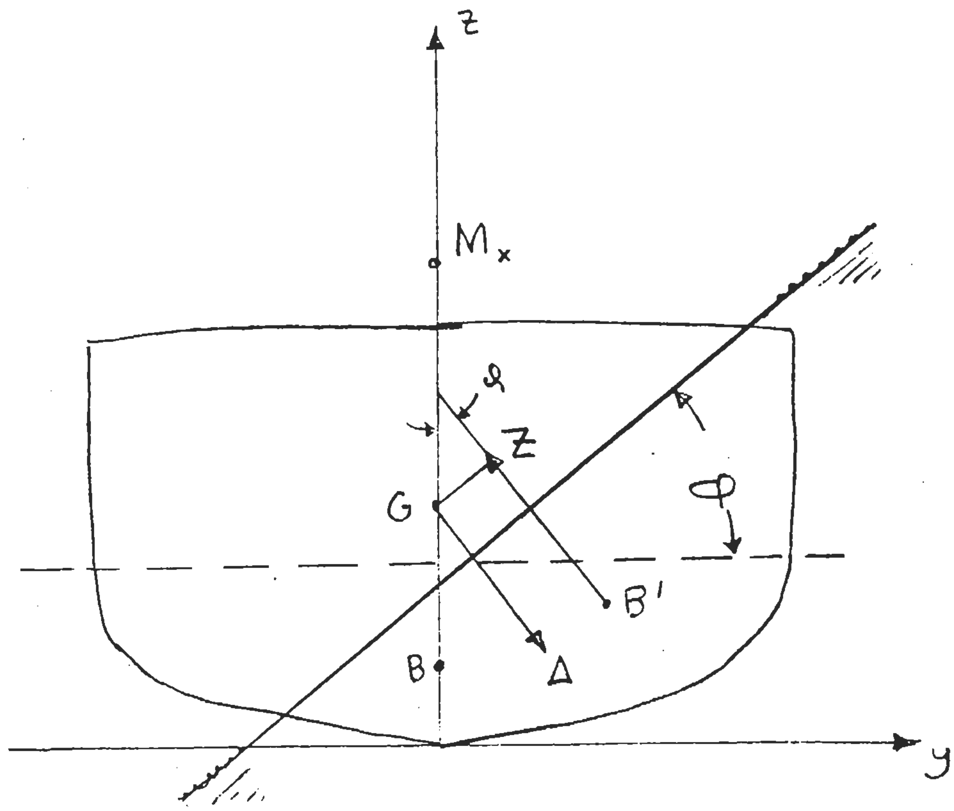
For the present discussion, we will consider the problem of finding the righting moment as a function of heel angle. As the heel angle increases, the immersed and emerged wedges of volume become more and more dissimilar. We can therefore no longer rely on waterplane properties alone to deduce the buoyant moment. As shown in the sketch below, all we can say is that the buoyant force moves from the initial point B to B'. The righting moment is therefore

$$M_x = -\Delta \bar{GZ} \tag{59}$$

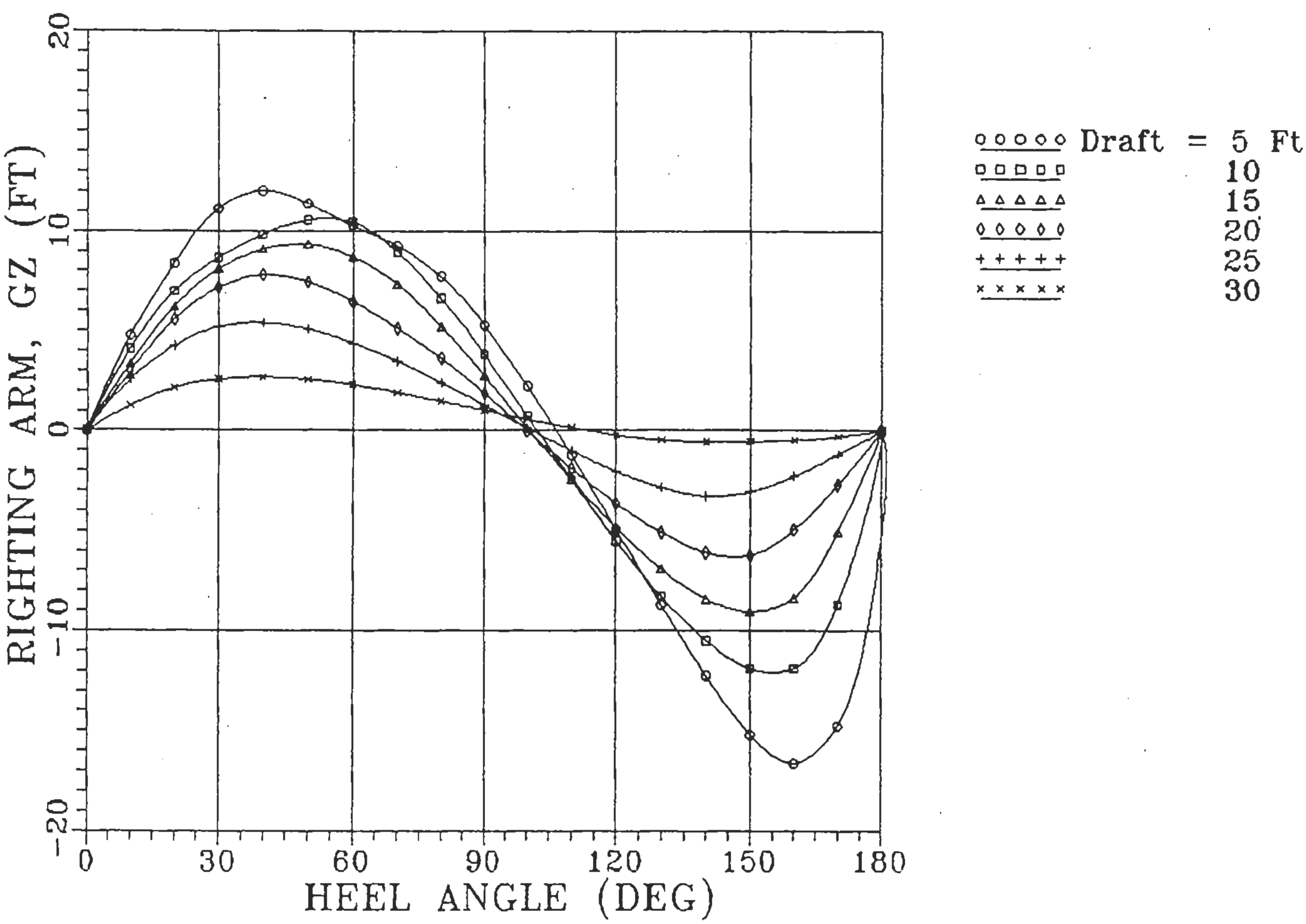
where the point Z is the intersection of a vertical line through the point B' and a horizontal line through G. The vertical and horizontal directions are, of course, taken with respect to the direction of gravity, not fixed relative to the body. The distance $\bar{GZ}$ is called the *righting arm*.

In the limit of small angles, we saw earlier that the bouyant force passes through the point M_x, which we called the transverse metacenter. However, at arbitrary heel angles, the bouyant force could pass above or below this point. In a way, the problem is conceptually simpler, since all we have to do is find the centroid of the submerged volume at a given heel angle. However, there is one hidden complication. We saw earlier that a floating body trims or heels about an axis through the center of floatation. However, this is only true in the limit of small angles. As the heel angle increases, the point of rotation moves. The actual point must be determined by requiring that the submerged volume remain constant.

This means that in addition to calculating the centroid of the assymetric submerged volume, one must adjust the height of the waterplane at each heel angle until the volume matches the initial upright volume to within a prescribed tolerance. In addition, for most shapes, the longitudinal position of the center of buoyancy changes with heel angle. In that case, an unbalanced trim moment exists, and the floating body must then trim until the moment is balanced.



Shown below are the calculated righting arms for our sample oceanographic ship for a range of initial drafts, and for an assumed height of the center of gravity of 20 feet. Notice that $\bar{GZ}(\phi)$ starts out fairly linear (the initial slope is just the metacentric height, $\bar{GM}$), but after about twenty degrees, the slope changes significantly. The maximum righting arms occur at around $\phi = 40$ degrees, and the stability then goes to zero in the neighborhood of 110 degrees. This angle is called the *range of stability*, since if the ship heeled over beyond this point (say due to a gust of wind or the impact of a big wave), it would capsize. The curves show that the ship is very stable upside down, at $\phi = 180$ degrees. However, since large ships are generally not designed to be watertight upside down, they will inevitably sink if they get in this predicament.

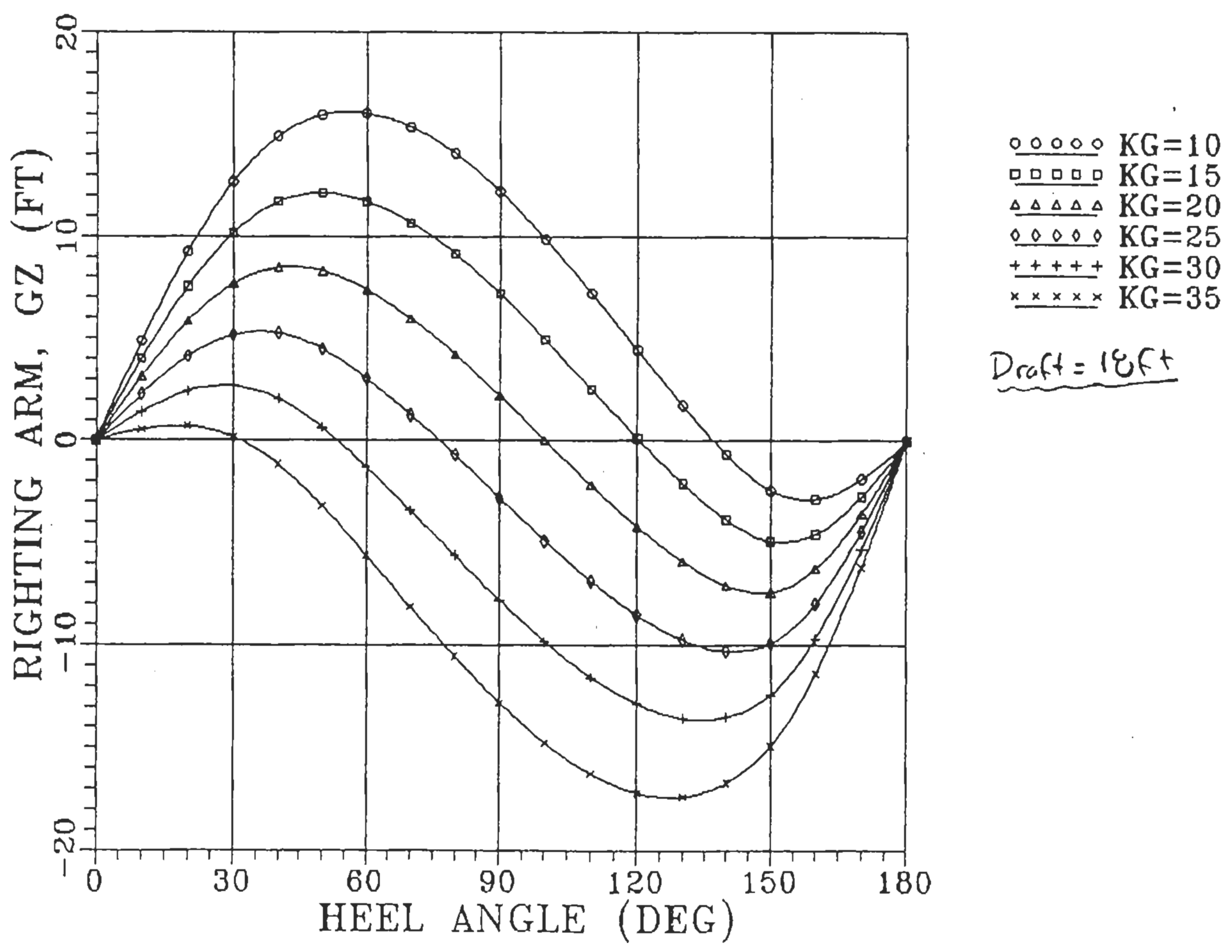


The righting arm curves on the preceding page were for one particular value of the height of the center of gravity. However, it is easy to see that if we shift G to a new point G', the righting arm will be changed by an amount

$$\bar{GZ}' = \bar{GZ} - \bar{GG'} \sin\phi \tag{60}$$

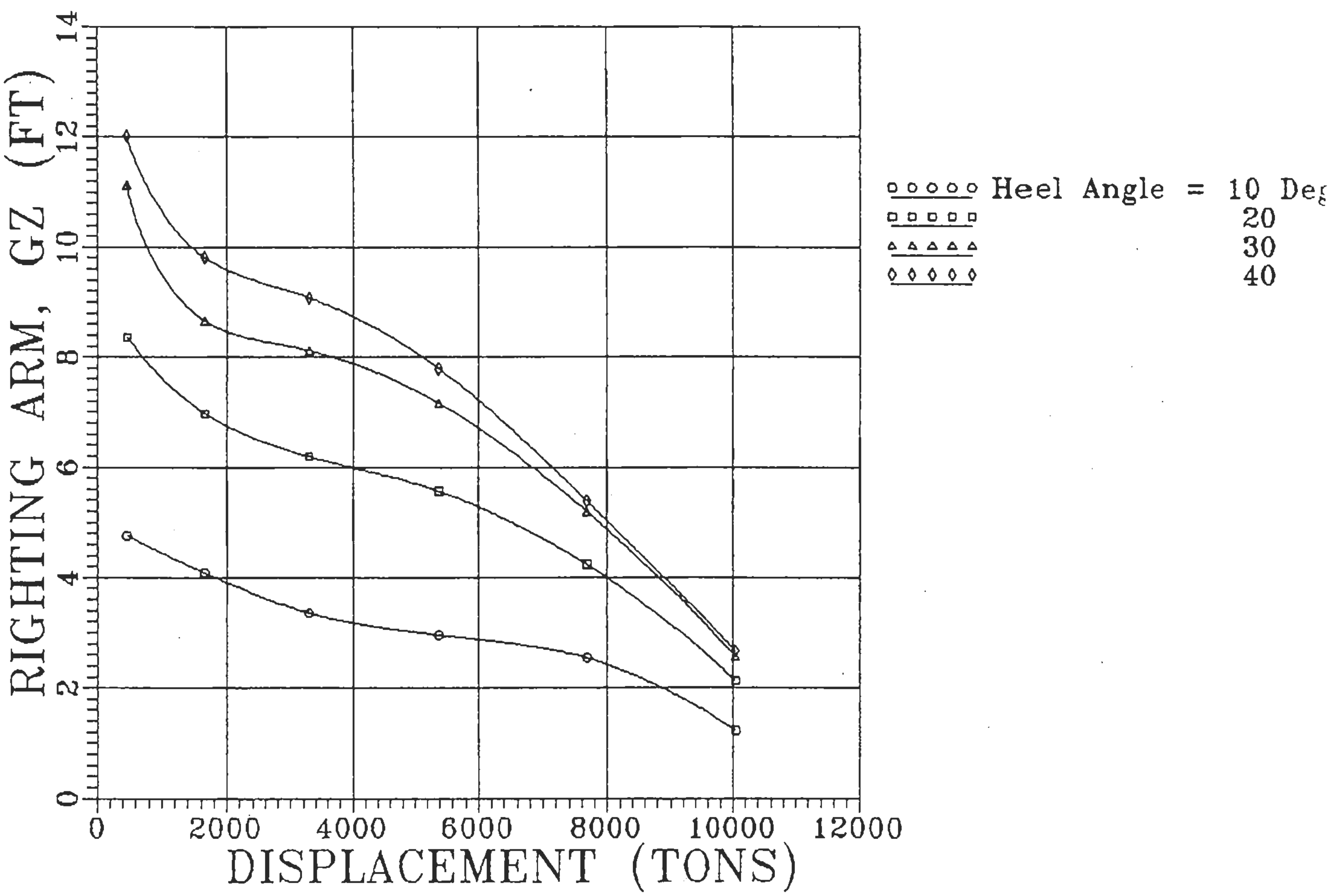
thus if G is raised, $\bar{GZ}$ is reduced, and vise-versa. This is illustrated below. Note that the range of stability is increased to around 140 degrees with the lowest center of gravity, while it is only 30 degrees for the highest value.

Since this correction for the height of the center of gravity does not affect the hydrostatic moment, the laborious calculation does not have to be repeated if the center of gravity is changed. One only needs to apply the simple correction formula above.



Another way of presenting righting arm data is to plot righting arm versus displacement for different values of heel angle, instead of righting arm versus heel angle for different values of displacement. These are called the ***cross curves of stability***, and are shown below (for the same data).

This obviously provides a convenient way to interpolate righting arm results at a particular displacement. They also served a useful purpose in the pre-computer era when balancing the hull at each heel angle to yield constant displacement. In that case, one simply calculated righting arms at a fixed heel angle for a set of assumed waterlines, and accepted whatever displacements came out. It didn't matter if they came out different at each heel angle, since the cross curves could be plotted in any event. In this case, however, trim balance was not necessarily achieved.



14 VARIOUS STABILITY TOPICS

14.1 The Inclining Experiment

It is important to know the exact height of the center of gravity of a floating object. In principle, it can be calculated from a careful tabulation of the weight and center of gravity of each part. However, many small boats and ships are built without a careful control of the exact position and weight everything that ends up on board. Even for large ships, where careful weight control is essential during design and construction, the predicted position of the center of gravity is subject to some uncertainty.

On the other hand, the position of the center of buoyancy and the metacenter can be calculated with fairly high accuracy if the shape of the hull is build accurately. In that case, one can derive the position of the center of gravity by applying a known heeling moment, measuring the heel angle, and then calculating the metacentric height from the small angle approximation

$$\delta M_x = \Delta \overline{GM_x} \delta\phi \tag{61}$$

Knowing both $\overline{GM_x}$ and the position of M_x, we can determine the position of G. This is frequently done when a ship is just built, before it is loaded with cargo and expendable supplies. The center of gravity can then be calculated at later times by allowing for the change due to the particular weights added.

Here is an example. The familiar oceanographic ship has just been launched, and is floating at a level draft of 16 ft. A 1000 pound weight, moved transversely a distance of 40 feet, produces a heel angle of 0.631 degrees. Find the height of the center of gravity.

From the curves of form, the displacement is 3682.2 *tons*. The metacentric height is therefore

$$\overline{GM_x} = \frac{\delta M_x}{\Delta\delta\phi} = \frac{1000 \times 40}{2240 \times 3682.2 \times 0.0631 \times 0.017453293} = 4.40\ ft \tag{62}$$

From the curves of form, we also see that the height of the transverse metacenter is 39.15 feet. The height of the center of gravity is therefore

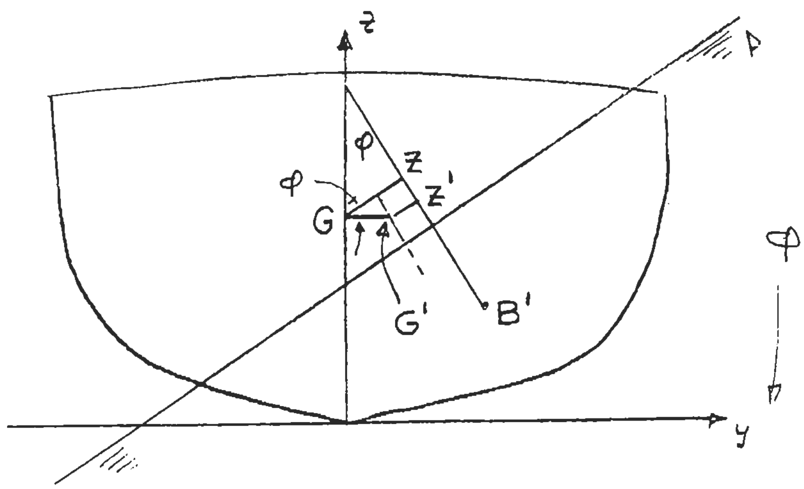
$$\overline{KG} = \overline{KM_x} - \overline{GM_x} = 39.15 - 4.40 = 34.75\ ft \tag{63}$$

Now suppose 2000 tons of equipment with a combined center of gravity of 12 feet above the base line are installed. The new center of gravity of the ship can now easily calculated. We leave this as an exercise ...

14.2 Transverse Shift in G

Even if a ship is symmetrical with respect to the xz plane, its center of gravity may be to one side due to the way it is loaded. This is frequently the case for sailboats, where the crew may move to one side to counterbalance the aerodynamic force on the sails. We already treated the small heel angle approximation in the example of the whale watching ship with the passengers moving out to one side. In the general case, if the center of gravity moves from the centerplane to a point G' which is a distance y_G out from the centerplane, the righting arm is changed by an amount

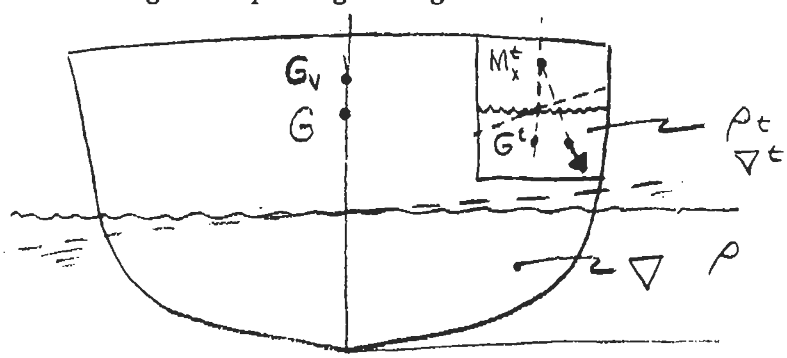
$$\overline{GZ}' = \overline{GZ} - y_G \cos\phi \tag{64}$$



14.3 Free Surface Effects

Ships and offshore platforms generally have tank compartments containing fluids such as fuel oil, drinking water or salt water ballast. If these tanks are completely full, the center of gravity of their contents does not change with heel angle. In that case, the fluids in these tanks act as though they were solids. On the other hand, if a tank is not completely full, it will have a free surface which remains horizontal as the vessel heels over. The center of gravity of the fluid is then not constant, but will change with heel angle.

This effect can be treated very simply if we again assume that the heel angle is small. As shown in the sketch, the gravitational force acting on the mass of fluid in the tank is directed along a line passing through the metacenter of the tank, M_x^t.



We can find the distance between the center of gravity of the fluid in the tank and the metacenter from

$$(\overline{GM_x})^t = \frac{\bar{I}_x^t}{\nabla^t} \tag{65}$$

where $\bar{I}_x^t$ is the moment of inertia of the free surface of fluid in the tank about an axis parallel to the x axis, but through its centroid, and ∇^t is the volume of fluid in the tank. This means that the *virtual* center of gravity of the tank (as far as small angular displacements are concerned) is at the point M_x^t. We can therefore imagine that the fluid in the tank is replaced by a solid point mass at its virtual center of gravity. This, in turn, raises the *virtual* center of gravity of the ship by an amount

$$\begin{aligned} \rho g \nabla \overline{GG_v} &= \rho_t g \nabla^t \overline{GM}_x^t \\ \overline{GG_v} &= \frac{\rho_t}{\rho} \frac{\bar{I}_x^t}{\nabla} \end{aligned} \tag{66}$$

where ρ_t is the mass density of the fluid in the tank. Note that the volume in the above equation is the volume of the ship, not the volume in the tank.

For example, suppose our sample oceanographic ship was floating at a level draft of 20 feet and had a 50 foot long, 30 foot wide tank on the deck. The tank was partially filled with sea water to provide a place to keep live specimens. What is the virtual rise in the center of gravity of the ship?

The transverse moment of inertia of the tank about its centroid is

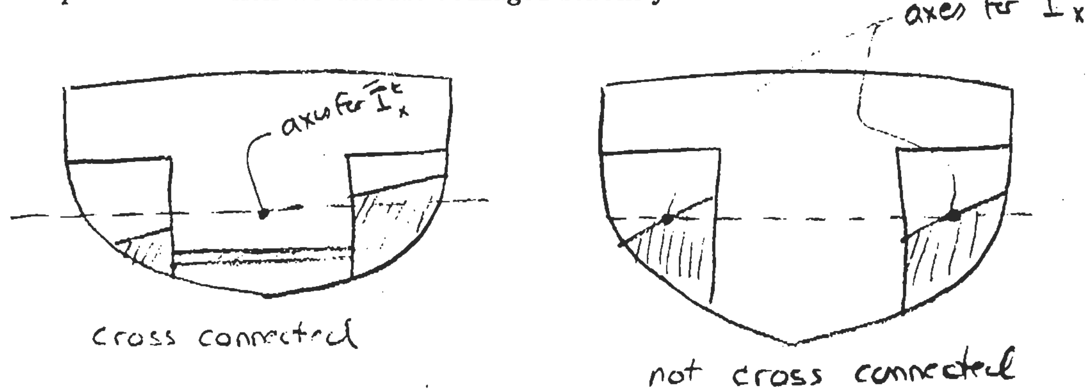
$$\bar{I}_x^t = \frac{1}{12} 50 \times 30^3 = 112,500 \ ft^4 \tag{67}$$

and the displaced volume of the ship is $5385.2 \times 35 = 188,482 \ ft^3$. Since the fluid in the tank has the same density as the fluid the ship is floating in, the density ratio is 1.0. The free surface correction is therefore

$$\overline{GG_v} = 112,500/188,482 = 0.598 \ ft \tag{68}$$

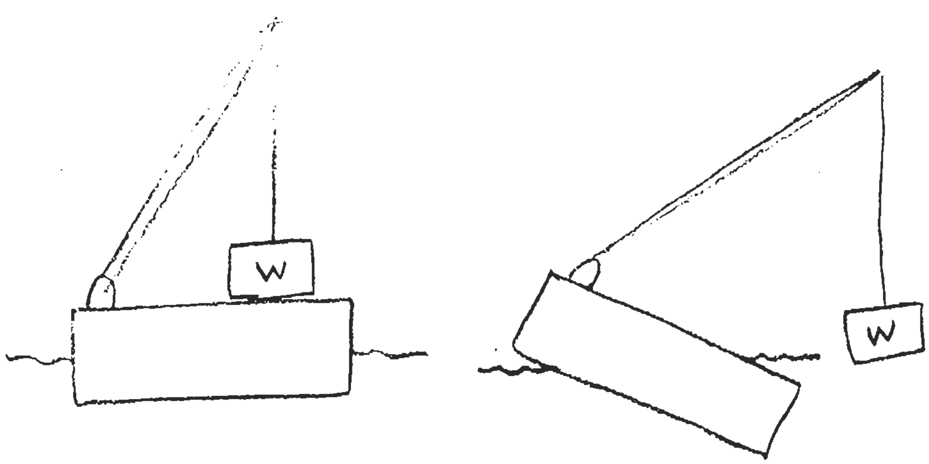
This is a significant increase in the height of the center of gravity, which might have an adverse effect on stability. Free surface corrections are therefore important. It is safer if tanks are either completely full or completely empty, but this is not always possible, since some fluids (such as fuel) need to be consumed during a voyage. If tanks are subdivided into smaller tanks, their free surface effect is greatly reduced since the sum of the moments of inertia of the subdivided tanks is much less than the moment of inertia of the original single tank.

The opposite situation exists if tanks on opposite sides of the ship are cross-connected, as shown in the sketch. These act like an inside out catamaran, so that the moment of inertia of the tank surface must be taken about an axis through the combined centroid of the two tanks. The free surface correction is therefore much higher than if the two tanks were isolated. Cross connecting tanks is sometimes done for convenience, for example so that fuel is consumed equally from tanks on both sides. But it is also done to increase safety after damage. We will return to this point later when we discuss damaged stability.



14.4 Hanging Weights

A hanging weight is similar to a free surface since the weight force acts through the suspension point, rather than through the initial center of gravity of the weight. This situation comes up frequently in offshore construction where heavy weights may be lifted by shipboard cranes. While the weight is resting on the deck, its virtual center of gravity is located at its physical center of gravity. As soon as the weight is lifted up an infinitesimal amount from the deck, its virtual center of gravity immediately moves to its suspension point. A ship which was initially stable could suddenly become unstable. It would then start to heel over and the weight would swing out to one side. This could spoil an otherwise good day.



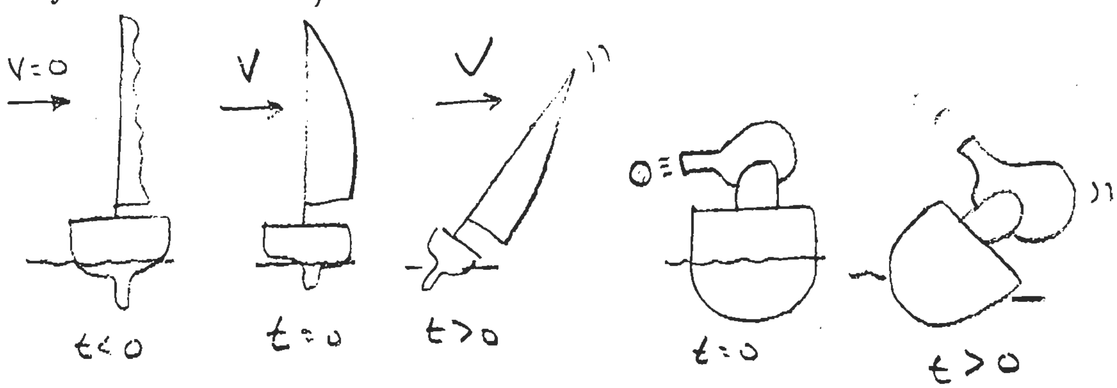
14.5 Running Aground

Speaking of bad days—suppose you ran aground on a sharp pointed rock and the tide went down. The buoyant volume of the hull would be reduced, and this would have to be balanced by the contact force of the rock acting on the bottom of the hull. If we assume that this force remains vertical as the ship heels over, this is equivalent to removing a weight of equal magnitude at the contact point. This again raises the height of the virtual center of gravity, thus reducing stability. The same situation occurs intentionally when a ship is launched or drydocked. To insure that these operations are safe, one must carefully calculate the grounding force and the corresponding virtual rise in the center of gravity during all stages of the process.

14.6 Dynamic Stability

Strictly speaking, hydrostatics is limited to the study of the equilibrium of objects at rest. However, there are situations involving motion of a floating body where the predominant force exerted on the body by the fluid is hydrostatic. Since the body may be moving, inertial forces and moments may act on the mass of the floating body.

An example is the response of an initially stationary, upright floating body to suddenly imposed heeling moment. This could be a sudden gust of wind hitting a sailboat or offshore platform, or an impulsive moment caused by firing a cannon (or maybe a water balloon).



At time $t = 0$ when the moment is applied, a heeling moment M_H exists, while the hydrostatic righting moment M_R is zero, since the object is initially upright. An unbalanced moment therefore exists, which will result in an angular acceleration. As the heel angle increases, the righting moment increases until it equals the heeling moment. At this point, the acceleration is zero, but since the floating object has developed an angular velocity (and hence kinetic energy), it will keep on going until the excess righting moment reduces the angular velocity to zero. Now the righting moment exceeds the heeling moment, and the object will now start rolling back in the opposite direction.

This oscillatory behavior would keep on indefinitely like a pendulum unless there were some form of energy dissipation. Of course, there always is, so that the object eventually settles down to the heel angle resulting in static equilibrium between the heeling and righting moment. The actual mechanism for energy dissipation, which includes both viscous losses and the radiation of waves, is beyond the scope of this subject. However, we can determine one very important quantity—what is the maximum possible oscillatory heel angle?

This can be done from a simple energy analysis. The work done by the heeling moment up to the angle ϕ_m is

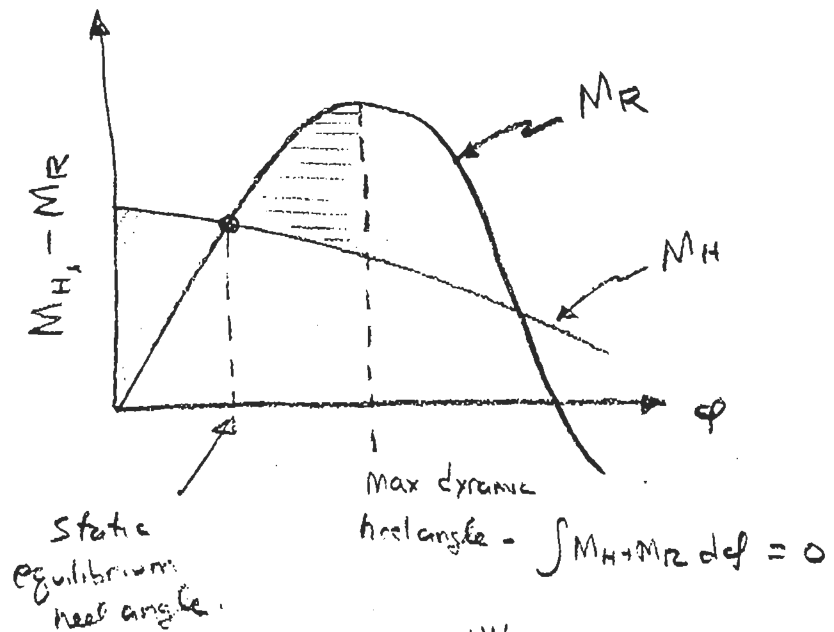
$$W_H = \int_0^{\phi_m} M_H(\phi)d\phi \tag{69}$$

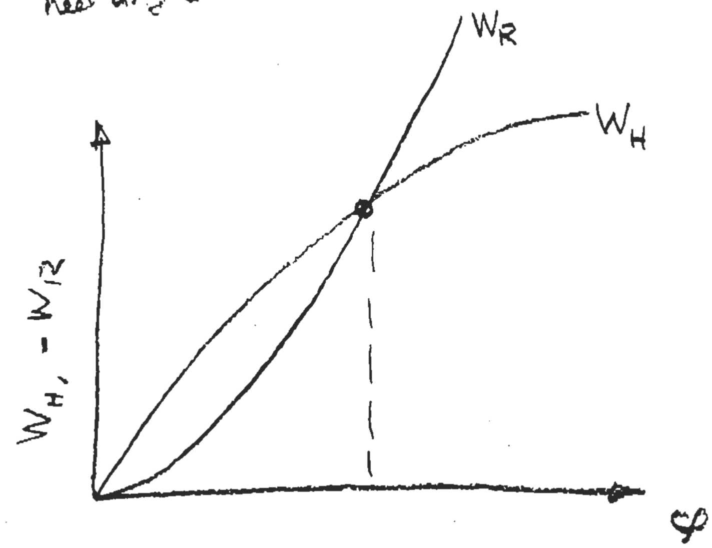
while the work done by the righting moment is

$$W_R = \int_0^{\phi_m} M_R(\phi)d\phi \tag{70}$$

The latter will generally be negative, since M_R will be negative if it is really a righting moment. The net work is the sum of the two, which must then equal the sum of the kinetic energy of the floating object plus the energy dissipated in waves and friction. If we ignore energy dissipation, then the maximum heel angle is the angle for which the sum of the work done by the heeling and righting moments is zero. This is illustrated in the plot below

If the maximum dynamic heel angle exceeds the range of statical stability, the object will capsize, even though the static equilibrium heel angle might be less than the range of stability. This is a safety concern, particularly for small ships, and regulations exist to insure that both the static and dynamic stability curves are adequate.





15 FLOODING—WHEN THINGS GO WRONG

While ships, offshore platforms and other floating devices are supposed to be watertight, they sometimes become partially flooded. The most common cause is collision, but structural failure, faulty operation, or even attacks by killer whales can result in unintentional flooding.

Sometimes flooding of a portion of a floating object is done intentionally. Submarines have ballast tanks which are flooded in order to reduce their buoyancy so that they will submerge. Floating drydocks flood and pump out compartments to turn themselves into floating elevators. Large offshore platforms, whose final draft may far exceed the water depth at the site of their construction, may be intentionally flooded during offshore assembly.

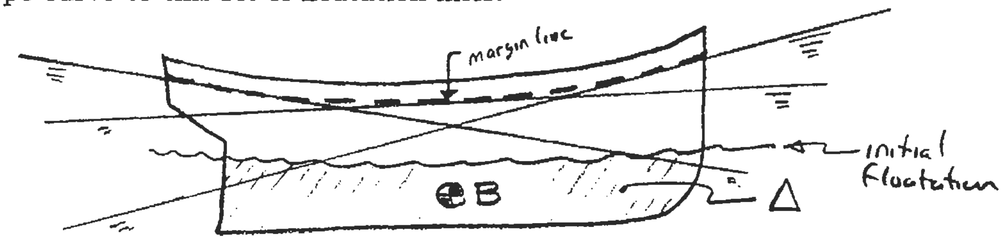
We will treat two kinds of flooding analysis. The first, which is called *floodable length*, consists of finding out how much of a floating object can be flooded without sinking. With this information, one can decide where to put watertight internal subdivision so that the ship or platform will be safe.

The other topic is to predict the floatation and stability after a prescribed set of internal compartments have been opened to the sea. We will look at this first as a small weight problem, and then treat the general case where the change in floatation may be large.

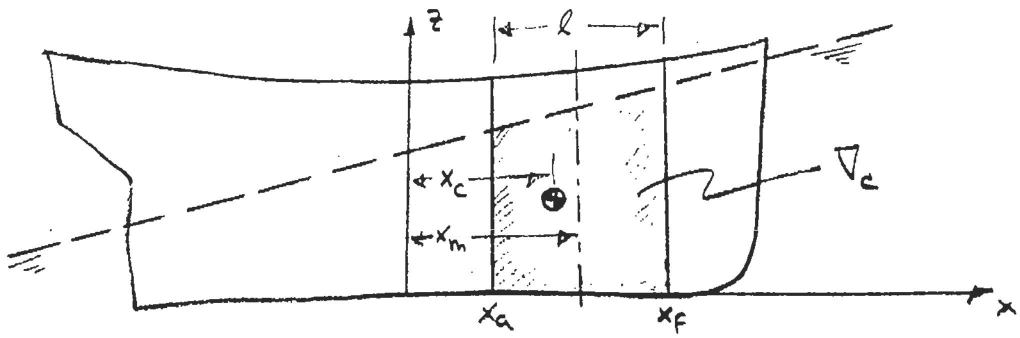
15.1 Floodable Length

We first define a *margin line* $z_m(x)$ as the maximum allowable height of the waterplane at any longitudinal position, x, occurring as a result of flooding. For ocean going ships, this line is defined by regulatory agencies to be at least 3 inches (76 mm) below the highest watertight deck. Regulations require that this point be even lower if the sheer (the difference in depth of the hull amidships and at the ends) is less than a prescribed amount. We will not cover these details here. The point is that we are told how much the hull is allowed to sink down in an emergency and still be considered safe.

Suppose the object were floating at a particular draft prior to damage. We therefore know its initial displacement, Δ and its longitudinal center of buoyancy, x_B. After flooding, there are an infinite number of possible floatations which will just keep the margin line dry. In fact, the margin line can be thought of as the envelope curve to this set of floatation lines.



Let us select one of these possible floatations, and find the length of flooded space within the hull that caused it. We will assume that the flooding is confined to the space in the hull between two planes $x = const$ located at x_f and x_a. The flooding within this region is assumed to be homogeneous, but that the volume of flooding is less than the volume of the space due to the presence of impermeable objects inside. This is expressed in terms of a ratio called the permeability, μ. If the volume of the flooded space is ∇_c, the volume of flooding is $\mu\nabla_c$ and the weight of the water entering the flooded compartment is $\rho g \mu \nabla_c$. As shown in the sketch below, the length of the flooded compartment is l and its mid point is $x_m = (x_f + x_a)/2$. The centroid of the volume of the water flooding the space is at x_c, which depends on the details of the shape. If the shape were prismatic (say, a barge), then the centroid of the flooding and the midpoint of the compartment would be the same.[1]



The value of the permeability, μ, depends on the contents of the region which is flooded. Generally one has neither the time or the detailed information necessary to

1 Also, if there were no trim.

compute an exact value. One therefore relies on statistical averages, which regulatory agencies generally prescribe. For example, empty compartments are generally assigned a value $\mu = 0.95$, indicating that on an average, five percent of the internal volume is occupied by structural elements, piping, wiring and other impermeable objects. Machinery spaces are assigned a permeability of $\mu = 0.85$, indicating that a somewhat larger part of the space is impermeable. Finally, cargo spaces have values which depend on the type of cargo being carried, and one can find tables for just about anything[1].

We can find ∇_c and x_c by writing down the force and moment equilibrium equations. Define ∇' as the volume of the hull underneath the flooded waterplane, and x'_B as the centroid of this volume. Then,

$$\begin{aligned} \mu\nabla_c &= \nabla' - \nabla \\ \nabla' x'_B &= \nabla x_B + \mu\nabla_c x_c \end{aligned} \tag{71}$$

Note that we are ignoring the contribution of $\overline{BG}\delta\theta$ to the moment, which is the assumption made earlier in developing the convenient moment per unit trim formula. This is only valid for ship–like forms.

Inverting the preceding equations gives the result

$$\begin{aligned} \nabla_c &= \frac{\nabla' - \nabla}{\mu} \\ x_c &= \frac{\nabla' x'_B - \nabla x_B}{\mu\nabla_c} \end{aligned} \tag{72}$$

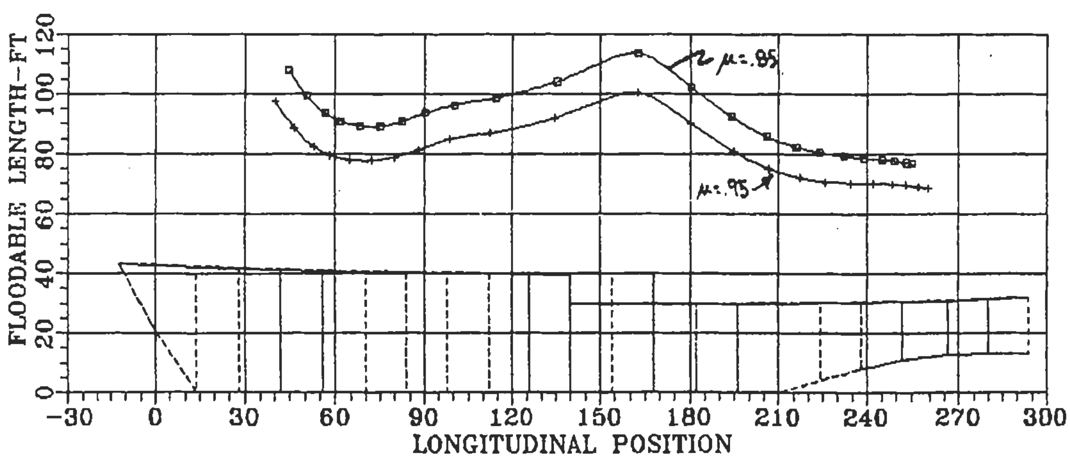
This is simple enough, but we still need to find x_m and l. Unless we are analyzing a prismatic shape, this requires an iterative process. There are a number of ways to do it, but here is one possible algorithm:

1. Assume $x_m = x_c$

[1] Grapefruit(in boxes) 46%, Onions(bags) 48%, Lanterns(cases) 80%, Tallow(barrels) 35%

2. Find $A_x(x_m)$ up to the flooded waterline
3. Estimate length from $l \approx \nabla_c/A_x$
4. Estimate the positions of the ends of the compartment; $x_f = x_m + l/2$, $x_a = x_m - l/2$.
5. Calculate a more exact value for ∇_c and x_c by dividing up the interval from x_f to x_a into a number of sub-intervals, find A_x for each one, and numerically integrate to obtain the volume and first moment.
6. These values will generally not be the same as the required values. Calculate the errors $\delta\nabla_c$ and δx_c.
7. Move the positions of x_f and x_a using the values of $A_x(x_f)$ and $A_x(x_a)$ in such a way as to reduce the error in volume and moment.
8. Repeat from step 5 until the errors are within a prescribed tolerance.

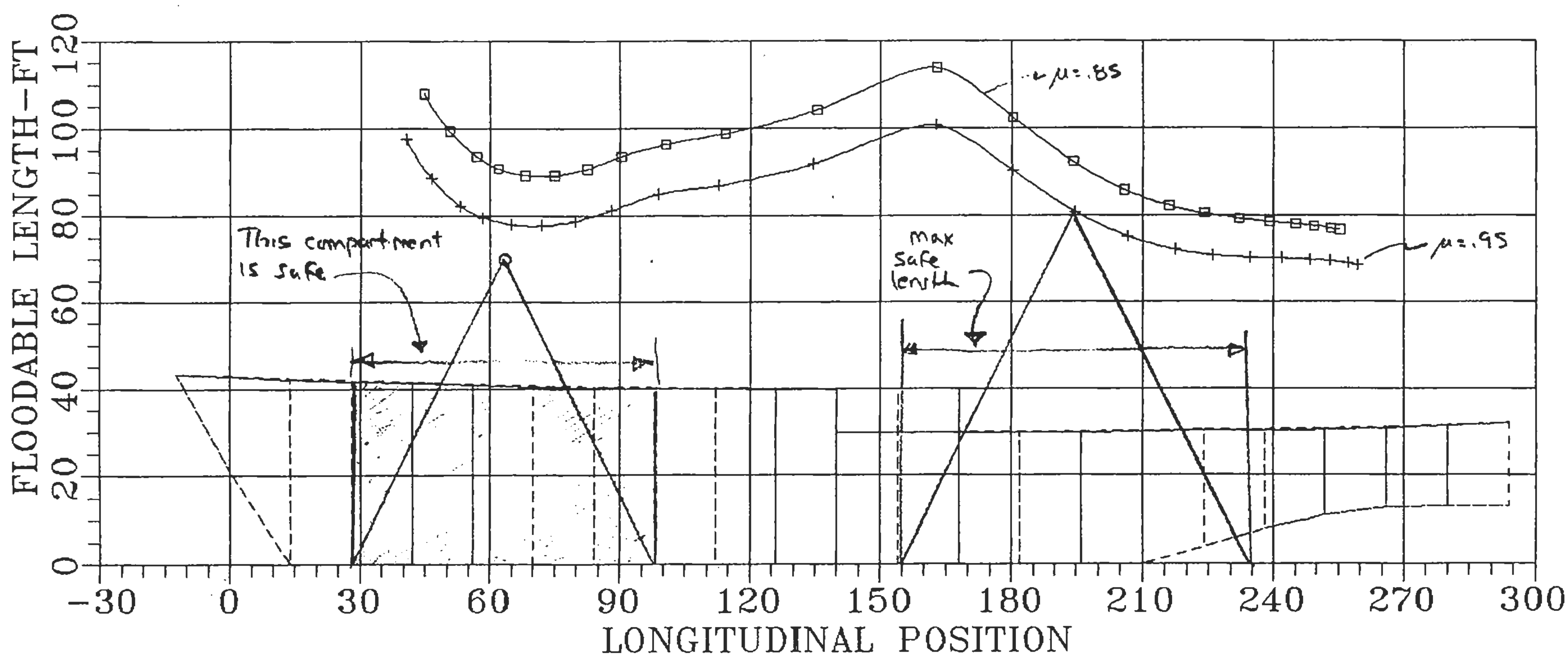
Now, this gives us the value of the floodable length, $l(x)$, corresponding to this one assumed floatation (these are sometimes called *trim lines*). Repeating the process for a systematically varied set of trim lines results in a floodable length curve, as shown in the figure for the sample oceanographic ship for an initial draft of 20 feet.



How can this information be used? A simple graphical interpretation of the result is shown below. If we start at any point on the floodable length curve and draw two lines with slope 2/1, the intercept with the x axis will be the forward and aft ends of the compartment which will just sink the hull down to the margin line. Conversely, if we start at the two ends of a given compartment, and draw lines upward with the same 2/1 slope, we can see whether or not the compartment is too long by checking whether their intersection is above or below the floodable length curve. If they intersect below the curve (corresponding to the appropriate permeability), we are safe, otherwise, we are not.

This brings up another point. The preceding discussion implies that we are safe if we can survive one compartment flooding. But, suppose some crazy runs into us, and hits right on a watertight bulkhead. In that case, two adjacent compartments will be flooded. Regulatory agencies therefore require that some types of ships have sufficient floodable length that two adjacent compartments can be flooded without submerging the margin line. But then, suppose that we run along side of an iceberg at high speed, and it rips open three or four compartments? In some cases, more than two-compartment subdivision may be required. However, as with any safety regulation, there is no such thing as *perfectly safe*. Those involved with safety standards must study the probability of occurrence of various hazards, and then come up with standards which are always a compromise between absolute safety and economic reality. One the standard is defined, however, a hydrostatic analysis can show with precision whether or not a given design meets the accepted level of safety.

OCEANOGRAPHIC RESEARCH SHIP PRELIMINARY DESIGN 20 FT DRAFT



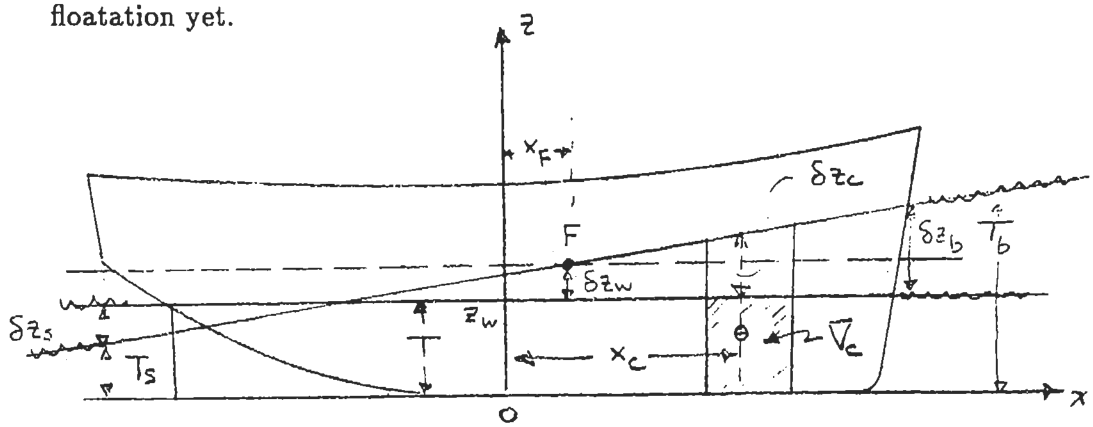
15.1 Small Flooding Approximations

We now want to estimate the final floatation after a prescribed length of the hull is flooded to the sea. However, we will assume here that the extent of flooding is small enough that we can make the same kind of assumptions that we did before in treating small weight additions.

Using the same notation as for floodable length, the flooded space extends from x_f to x_a, with centroid at x_c. We want to find the final floatation, which can either be expressed as the sinkage, δz_w and trim, t, or the final drafts T_b and T_s at the bow and stern. This is the inverse of the floodable length problem, where the final floatation was given, and the extent of the flooding had to be found.

There are two ways to analyze this problem; the *Added Weight Method* and the *Lost Buoyancy Method.* Both give the same final answer, are equally correct, but very different conceptually.

The added weight method starts with the ship floating at initial draft T, or equivalently, with the waterplane at $z = z_w$. If the compartment is opened to the sea, a weight $w_{fz} = \rho g \mu \nabla_c$ of sea water will come in. Note that ∇_c is the volume of the flooded space *below the initial waterplane*, not up to the final waterplane as in the floodable length analysis. This is just as well, since we do not know the final floatation yet.



The quantity w_{fz} is our initial guess for the added weight. With this value of the weight, we can use the good old small weight method to find the change in draft and the trim based on the tons per inch and the moment to trim one inch,

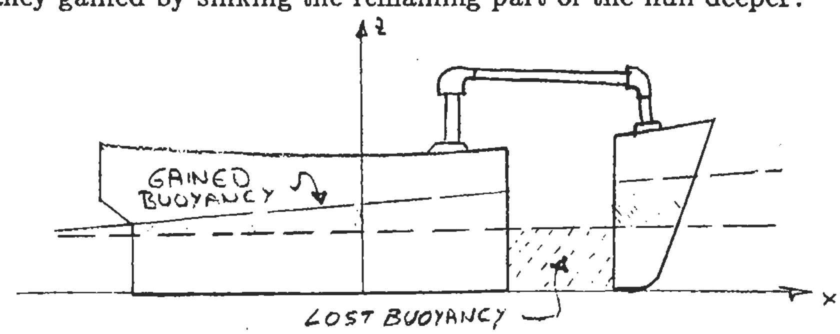
$$
\begin{aligned}
\delta z_w &= w_{fz}/(12 \times TPI) \\
t &= w_{fz} \times (x_c - x_F)/(12 \times MTI) \\
\delta z_b &= \delta z_w + (\mathcal{L}/2 - x_F) \times t/\mathcal{L} \\
\delta z_s &= \delta z_w - (\mathcal{L}/2 + x_F) \times t/\mathcal{L} \\
\delta z_c &= \delta z_w + (x_c - x_F) \times t/\mathcal{L}
\end{aligned} \tag{73}
$$

where δz_c is the increment in draft calculated at the centroid of the damaged space.

Now here comes the problem. Since the waterplane at the location of the flooding has increased, more water will come in. If the waterplane area of the damaged compartment (at the initial floatation) is A_c, the increment of added weight will be approximately $\delta w_f = \rho g \mu A_c \delta z_c$, so that the total added weight will be $w_f = w_{fz} + \rho g \mu A_c \delta z_c$. We can therefore compute our next estimate of the final floatation by replacing w_{fz} with w_f in equation 73.

As a result, the hull sinks down more, thus more water comes in, causing it to sink even more. Fortunately, the process converges to some limiting value, which is the answer that we are looking for. This is illustrated in the sample FORTRAN program listed in this section. In this example, the sample oceanographic ship is assumed to be damaged at an initial draft of 20 feet. The weight of the water flooding the damaged space up to the initial waterplane is assumed to be 397 tons, located with centroid 50 feet forward of amidships. The waterplane area of the damaged space is given as 965 ft^2. All other quantities are taken from the tabulated hydrostatic data that we have seen before. Note that the result is converged to 0.001 ft of draft after six iterations.

We will now look at the lost buoyancy approach. In this case, the flooded region is no longer considered to be part of the hull. As shown in the sketch, it is though the flooded part had been sawed off and thrown away, and the two remaining halves mechanically connected in some way. No weight is added, since the water inside the flooded compartment is just part of the ocean. The displacement of the hull is therefore the same, before and after the damage. The buoyancy must also be the same. This is accomplished by replacing the *lost buoyancy* in the flooded space with buoyancy gained by sinking the remaining part of the hull deeper.



Similarly, the longitudinal position of the center of gravity of the hull is unchanged, so that the moment about the y axis of the lost buoyancy must equal the moment of the gained buoyancy.

Proceeding with the details of the solution, the lost buoyancy is $\rho g \mu \nabla_c$, which is the same as our first approximation to the added weight in the added weight method, w_{fz}. To get the gained buoyancy, we first need to get the tons per inch, center of floatation and moment to trim one inch of our newly remodelled ship,

$$\begin{aligned}
A_z &= 420 \times TPI \\
A_{zd} &= (A_z - \mu \times A_c) \\
TPI_d &= A_{zd}/420 \\
x_{Fd} &= (A_z \times x_F - \mu \times A_c \times x_c)/(A_z - \mu \times A_c) \\
I_y &= 420 \times MTI \times \mathcal{L} \\
I_{yd} &= I_y - \mu \times A_c \times (x_c - x_F)^2 \\
\bar{I}_{yd} &= I_{yd} - Azd \times (x_{Fd} - x_F)^2 \\
MTI_d &= \bar{I}_{yd}/(420 \times \mathcal{L})
\end{aligned} \qquad (74)$$

We now have all the necessary characteristics of the hull after damage. While this was more work that in the added weight method, we can now solve directly for the sinkage and trim without any iteration,

$$\begin{aligned}
\delta z_w &= w_{fz}/(12 \times TPI) \\
t &= w_{fz} \times (x_c - x_{Fd})/(12 \times MTI) \\
\delta z_b &= \delta z_w + (\mathcal{L}/2 - x_{Fd}) \times t/\mathcal{L} \\
\delta z_s &= \delta z_w - (\mathcal{L}/2 + x_{Fd}) \times t/\mathcal{L}
\end{aligned} \qquad (75)$$

The lost buoyancy method is also included in the program shown in this section. Believe it or not, the final drafts at the bow and stern are exactly the same. However, intermediate quantities are not. For example, the draft increment, δx_w is 0.996 ft for the added weight method and 0.941 ft for the lost buoyancy method. This might seem like a mistake, but the explanation is that these both represent the draft increment at the center of floatation, and the centers of floatation are different in the two methods. In the first case, it is in the same position as for the intact hull, while in the second case, it has moved to the centroid of the new waterplane which has excluded the damaged part.

One final comment. Since the lost buoyancy method did not require an iterative solution, there is no reason why the added weigh method should. In fact, one could solve algebraically for the unknown final drafts, which would then result in a set of equations which look like the lost buoyancy method. The iterative added weight method is conceptually simpler, and also allows for the possibility of refinement to allow for larger floatation changes.

The following FORTRAN program computes the floatation after damage using both the added weight and lost buoyancy methods. The parameter statement sets up the data for the calculation for the oceanographic ship discussed in this section. The permeability is assumed to be one in this example.

```
      REAL*4 MTI,LPP,IY,IYD,IYDF,MTID
      PARAMETER( WFZ=397.0, AC=965.0, TPI=37.45, MTI=675.81 )
      PARAMETER( LPP=294.0, XF=-16.72, XC=50.0 )
****** ADDED WEIGHT METHOD ITERATIVE SOLUTION **************************
      WRITE(*,'(A)')  '                ADDED WEIGHT ITERATIVE METHOD '
      WRITE(*,'(A)') ' ----WF--- ----DZ--- ----T---- ---DZC--- ---DZB---
     * ---DZS--- '
      WF=WFZ
      DO 1 N=1,6
         DZ=WF/(12.0*TPI)
         T=WF*(XC-XF)/(12.0*MTI)
         DZC=DZ+(XC-XF)*T/LPP
         WF=WFZ+AC*DZC/35.0
         DZB=DZ+(LPP/2.0-XF)*T/LPP
         DZS=DZ-(LPP/2.0+XF)*T/LPP
         WRITE(*,'(6F10.3)') WF,DZ,T,DZC,DZB,DZS
 1    CONTINUE
****** LOST BUOYANCY METHOD ********************************************
      AZ=420.0*TPI
      AZD=AZ-AC
      TPID=AZD/420.0
      IY=420.0*LPP*MTI
      IYD=IY-AC*(XC-XF)**2
      XFD=(AZ*XF-AC*XC)/(AZ-AC)
      IYDF=IYD-AZD*(XFD-XF)**2
```

```
      MTID=IYDF/(420.0*LPP)
      DZ=WFZ/(12.0*TPID)
      T=WFZ*(XC-XFD)/(12.0*MTID)
      DZB=DZ+(LPP/2.0-XFD)*T/LPP
      DZS=DZ-(LPP/2.0+XFD)*T/LPP
      WRITE(*,'(A)') '                    LOST BUOYANCY METHOD          '
      WRITE(*,'(A)') ' ---TPID-- ---MTID-- ---XFD--- ----DZ--- ----T----
     * ---DZB--- ---DZS---- '
      WRITE(*,'(7F10.3)') TPID,MTID,XFD,DZ,T,DZB,DZS
      STOP
      END
```

```
                ADDED WEIGHT ITERATIVE METHOD
----WF--- ----DZ--- ----T---- ---DZC--- ---DZB--- ---DZS---
  441.793      .883     3.266     1.625     2.702     -.564
  446.847      .983     3.635     1.808     3.007     -.628
  447.417      .994     3.676     1.829     3.042     -.635
  447.482      .996     3.681     1.831     3.045     -.636
  447.489      .996     3.682     1.831     3.046     -.636
  447.490      .996     3.682     1.831     3.046     -.636
                LOST BUOYANCY METHOD
---TPID-- ---MTID-- ---XFD--- ----DZ--- ----T---- ---DZB--- ---DZS----
   35.152   638.747   -21.081      .941     3.682     3.046     -.636
Stop - Program terminated.
```

15.2 Damage Stability

Sinking is not the only concern if a floating object is partially flooded due to damage. It is possible that transverse stability may be sufficiently reduced to cause capsizing. Having solved for the new foatation, it is simple to estimate the metacentric height after damage, which provides with an indication of initial stability. Large angle stability after damage requires brute force calculations which are conceptually not any different from the case of intact stability. In this section we will develop expressions for the metacentric height after damage, again using the added weight and lost buoyancy method.

In addition to the quantities used before to determine sinkage and trim, we need to know the height of the centroid of the flooded space (up to the initial waterplane), z_c, and the moment of inertia of the initial damaged waterplane about the x axis, I_c. To simplify the development, we will assume that the permeability, μ is one.

Following the added weight approach, we first determine the height of the centroid of the final added weight,

$$z_{cf} = \frac{z_c \times w_{fz} + z_w \times A_c \times \delta z_c \times \rho g}{w_f} \tag{76}$$

Note that the height of the additional flooding (beyond the initial waterplane) is taken to be at the height of the initial waterplane, z_z, which is consistent with a small weight approximation. We now know both the added weight and its vertical center, so that we can now calculate the new center of gravity of the ship

$$\overline{KG}_f = \frac{\overline{KG}\Delta + z_{cf} \times w_f}{\Delta + w_f} \tag{77}$$

If we consider the flooded water to be an added weight, it must act like water in a tank, so that there must be a free surface correction. The virtual height of the center of gravity is therefore

$$\overline{KG}_v = \overline{KG}_f + \rho g I_c/(\Delta + w_f) \tag{78}$$

The height of the center of buoyancy is also changed, since some floatation has been added (approximately) at the initial waterplane,

$$\overline{KB}_f = \frac{\Delta \times \overline{KB} + w_f \times z_w}{\Delta + w_f} \tag{79}$$

The transverse metacentric radius $\overline{BM}_f$ after flooding will also be different. This is the moment of inertia of the waterplane divided by the displaced volume. The waterplane inertia is (approximately) unchanged, but the displaced volume has increased. We can therefore obtain the metacentric radius after flooding from its intact value by multiplying by the ratio of displacements,

$$\overline{BM}_f = \overline{BM}\frac{\Delta}{\Delta + w_f} \tag{80}$$

The final metacentric height after flooding is

$$\overline{GM}_f = \overline{KB}_f + \overline{BM}_f - \overline{KG}_v \tag{81}$$

Using the lost buoyancy approach, the height of the center of gravity is unchanged (since no weight was added) and there is no free surface correction, since the water in the flooded space is part of the ocean. However, the height of the center of buoyancy has changed, but by an amount which is different from the added weight method. In this case, buoyancy has been moved from the centroid of the flooded compartment (the lost buoyancy) to the initial waterplane,

$$\overline{KB}_f = \frac{\Delta \times \overline{KB} + w_f \times (z_w - z_c)}{\Delta} \tag{82}$$

The transverse metacentric radius is also different, but in this case it is because the waterplane inertia of the flooded space has been lost, while the total displacement remains the same.

$$\overline{BM}_f = \rho g \frac{I_x - I_c}{\Delta} \tag{83}$$

The metacentric height, as before, is obtained from the difference in height of B and G.

The following computer program performs these calculations, continuing from the preceding example. The results show that the metacentric height according to the added weight method is 4.827 ft, while according to the lost buoyancy method it is 5.228 ft. This, again, would seem like a mistake until we realize that the righting moment is the physical quantity which should be the same, regardless of the method used. Recalling that $\delta M_x = \Delta \overline{GM_x} \delta\phi$, the product of metacentric height and displacement should be the same. Since the displacements are different in the two cases (in the added weight case it is increased by the weight of the flooding, in the lost buoyancy case it stays the same), the metacentric heights have to be different. The example verifies that their product is the same with both methods.

Note that the product of metacentric height and displacement of the intact ship is 31,288. The stability is therefore about ten percent less as a result of the flooding. This happens not to be a very major difference in this case, but this is not always the case.

```
      REAL*4 KG,KB,KM,IC,KGF,KGV,KBF,KBL,IXL
      PARAMETER( KG=32.0, KB=12.64, KM=37.81, IC=208333.0 )
      PARAMETER( AC=965.0, DZC=1.831, DELTA=5385.2, WF=447.49)
      PARAMETER( WFZ=397.0, ZC=12.9, ZW=20 )
      RHOG=1.0/35.0

****** ADDED WEIGHT SOLUTION FOR DAMAGE STABILITY **********************
      ZCF=(ZC*WFZ+ZW*AC*DZC*RHOG)/WF
      DELTAF=DELTA+WF
      KGF=(KG*DELTA+ZCF*WF)/DELTAF
      FSC=IC*RHOG/DELTAF
      KGV=KGF+FSC
      KBF=(DELTA*KB+WF*ZW)/DELTAF
      BM=KM-KB
      BMF=BM*DELTA/DELTAF
      GMF=KBF+BMF-KGV
      DELGMF=DELTAF*GMF
      WRITE(*,'(''  ADDED WEIGHT: GM='',F8.3,''  GM*DISP='',F10.0)')
     *          GMF,DELGMF

****** LOST BUOYANCY SOLUTION FOR DAMAGE STABILITY *********************
      KBL=(DELTA*KB+WFZ*(ZW-ZC))/DELTA
      IXL=BM*DELTA/RHOG-IC
      BML=IXL*RHOG/DELTA
      GML=KBL+BML-KG
      DELGML=DELTA*GML
      WRITE(*,'('' LOST BUOYANCY: GM='',F8.3,''  GM*DISP='',F10.0)')
     *          GML,DELGML
      STOP
      END
```

```
 ADDED WEIGHT: GM=   4.827  GM*DISP=    28154.
LOST BUOYANCY: GM=   5.228  GM*DISP=    28154.
Stop - Program terminated.
```