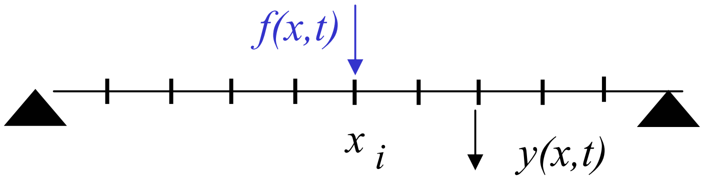
Introduction to Numerical Analysis for Engineers

- Systems of Linear Equations — Mathews
  - Cramer's Rule
  - Gaussian Elimination — 3.3-3.5
    - Numerical implementation
      3.3-3.4
    - Numerical stability
      - Partial Pivoting
      - Equilibration
      - Full Pivoting
    - Multiple right hand sides
    - Computation count
    - LU factorization — 3.5
    - Error Analysis for Linear Systems — 3.4
      - Condition Number
    - Special Matrices
  - Iterative Methods — 3.6
    - Jacobi's method
    - Gauss-Seidel iteration
    - Convergence

Linear Systems of Equations
Tri-diagonal Systems

Forced Vibration of a String



Harmonic excitation

$$f(x,t) = f(x)\cos(\omega t)$$

Differential Equation

$$\frac{d^2y}{dx^2} + k^2 y = f(x)$$

Boundary Conditions

$$y(0) = 0\ , \quad y(L) = 0$$

Finite Difference

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} \simeq \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Discrete Difference Equations

$$y_{i-1} + \left((kh)^2 - 2\right) y_i - y_{i+1} = f(x_i)h^2$$

Matrix Form

$$\begin{bmatrix} (kh)^2-2 & 1 & \cdot & \cdot & \cdot\ \cdot & 0 \\ 1 & (kh)^2-2 & 1 & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & 1 & (kh)^2-2 & 1 & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot\ 1 & (kh)^2-2 \end{bmatrix} \overline{\mathbf{x}} = \begin{Bmatrix} f(x_1)h^2 \\ \cdot \\ \cdot \\ f(x_i)h^2 \\ \cdot \\ \cdot \\ f(x_n)h^2 \end{Bmatrix}$$

Tridiagonal Matrix

$kh < 1$ Symmetric, positive definite: No pivoting needed

Linear Systems of Equations
Tri-diagonal Systems

General Tri-diagonal Systems

$$\begin{bmatrix} a_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ b_2 & a_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & b_i & a_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & b_n & a_n \end{bmatrix} \overline{\mathbf{x}} = \begin{Bmatrix} f_1 \\ \cdot \\ \cdot \\ f_i \\ \cdot \\ \cdot \\ f_n \end{Bmatrix}$$

$$\overline{\overline{\mathbf{L}}} = \begin{bmatrix} 1 & & \cdot & \cdot & \cdot & \cdot & 0 \\ \beta_2 & 1 & & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & \beta_i & 1 & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \beta_n & 1 \end{bmatrix}$$

LU Factorization

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{L}\mathbf{U}}}$$

$$\overline{\overline{\mathbf{U}}} = \begin{bmatrix} \alpha_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ & \alpha_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \alpha_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \alpha_n \end{bmatrix}$$

$$\begin{aligned} \overline{\overline{\mathbf{L}}}\overline{\mathbf{y}} &= \overline{\mathbf{f}} \\ \overline{\overline{\mathbf{U}}}\overline{\mathbf{x}} &= \overline{\mathbf{y}} \end{aligned}$$

Linear Systems of Equations
Tri-diagonal Systems

LU Factorization

$$\overline{\overline{\mathbf{L}}} = \begin{bmatrix} 1 & & \cdot & \cdot & \cdot & \cdot & 0 \\ \beta_2 & 1 & & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & \beta_i & 1 & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \beta_n & 1 \end{bmatrix}$$

$$\overline{\overline{\mathbf{U}}} = \begin{bmatrix} \alpha_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ & \alpha_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \alpha_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \alpha_n \end{bmatrix}$$

Reduction

$$\alpha_1 = a_1$$

$$\beta_k = \frac{b_k}{\alpha_{k-1}} \; , \quad \alpha_k = a_k - \beta_k c_{k-1} \; , \; k = 2, 3, \ldots n$$

Forward Substitution

$$y_1 = f_1 \; , \quad y_i = f_i - \beta_i y_{i-1} \; , \; i = 2, 3, \ldots n$$

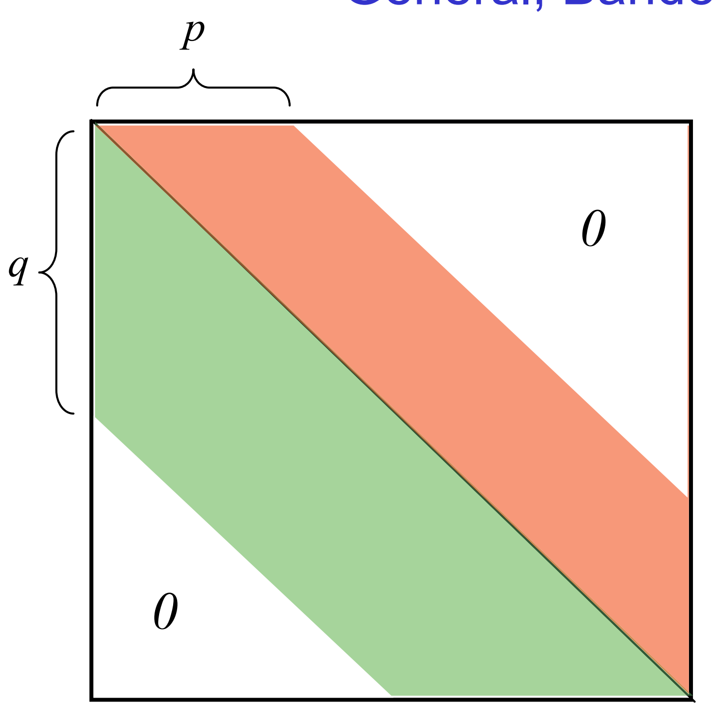
Back Substitution

$$x_n = \frac{y_n}{\alpha_n} \; , \quad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_1} \; , \; i = n-1, \ldots 1$$

LU Factorization:	2*(n-1) operations
Forward substitution:	n-1 operations
Back substitution:	n-1 operations
Total:	4(n-1) ~ O(n) operations

Linear Systems of Equations
Special Matrices

General, Banded Coefficient Matrix



$$\left.\begin{array}{l} j > i + p \\ i > j + q \end{array}\right\} a_{ij} = 0$$

p super-diagonals
q sub-diagonals
$w = p+q+1$ bandwidth

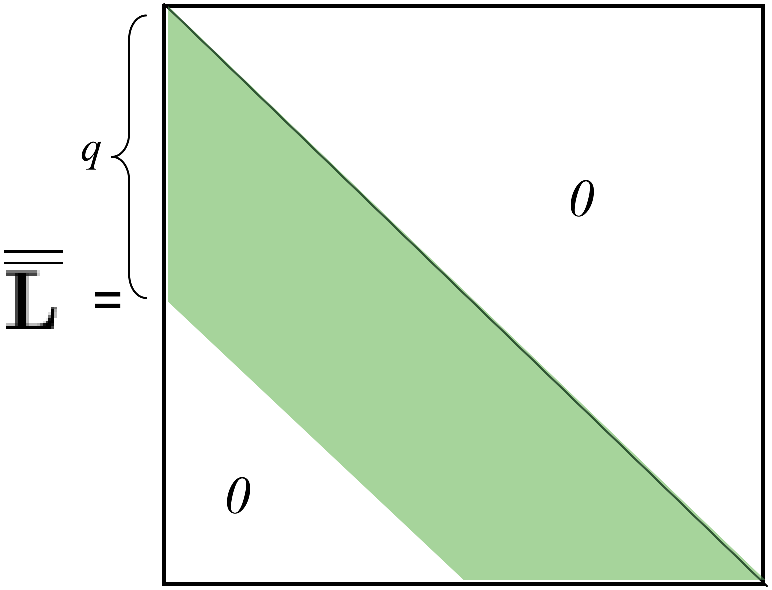
Banded Symmetric Matrix

$$\begin{aligned} a_{ij} &= a_{ji}\ , \quad |i-j| \leq b \\ a_{ij} &= a_{ji} = 0\ , \quad |i-j| > b \end{aligned}$$

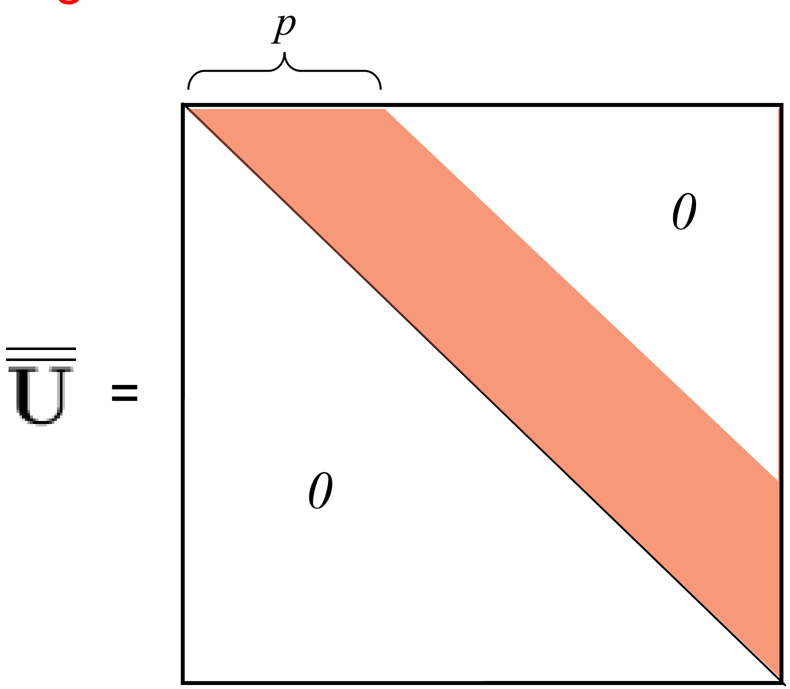
b is half-bandwidth

Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Gaussian Elimination
No Pivoting



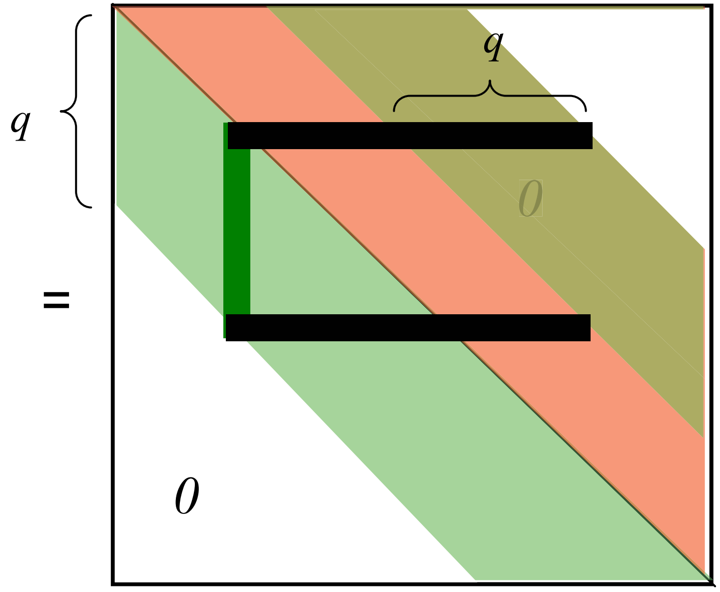
$$m_{ij} = 0,\ j > i,\ i > j + q$$



$$u_{ij} = 0,\ i > j,\ j > i + p$$

Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Gaussian Elimination
With Pivoting

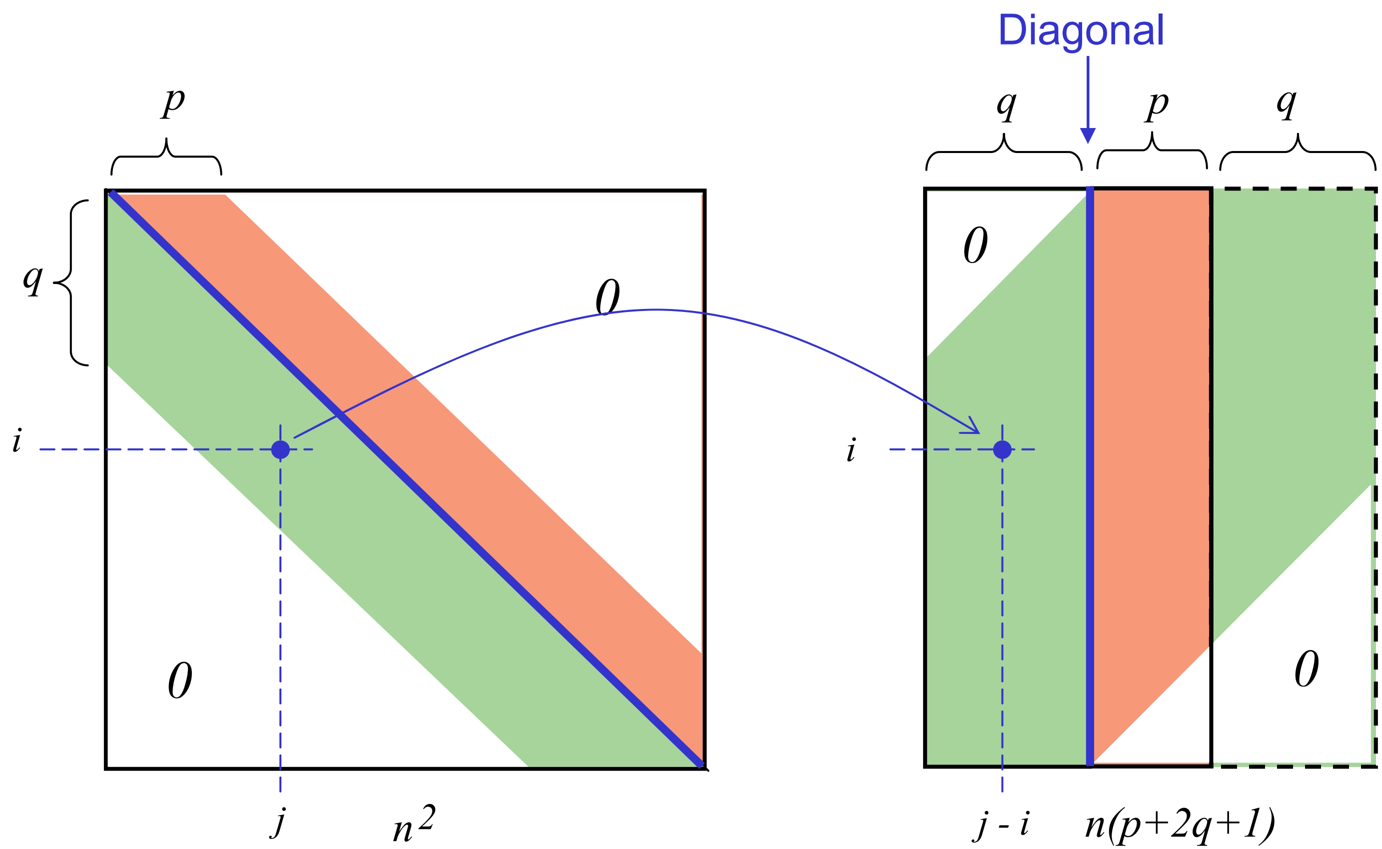


$$m_{ij} = 0, \ j > i, \ i > j + q$$

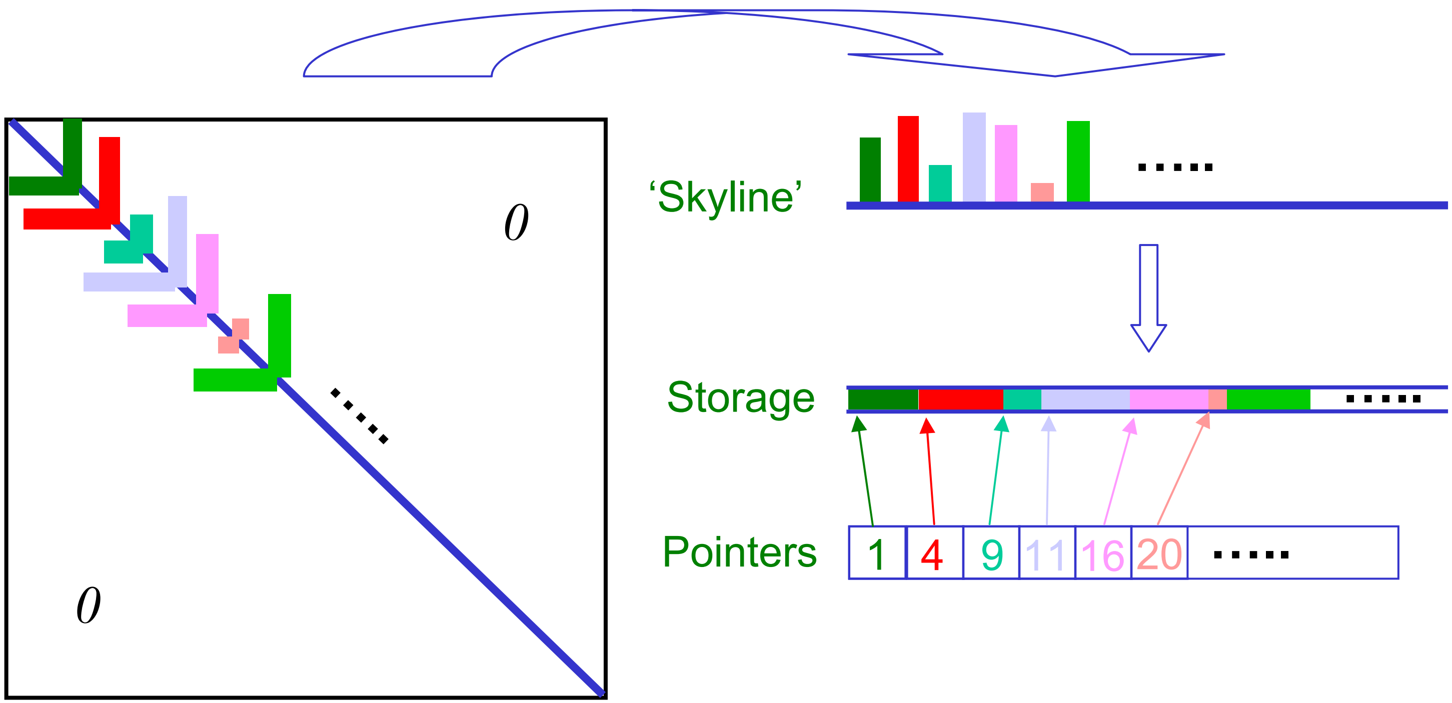
$$u_{ij} = 0, \ i > j, \ j > i + p + q$$

Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Compact Storage



Sparse and Banded Coefficient Matrix
'Skyline' Systems



Skyline storage applicable when no pivoting is needed, e.g. for banded, symmetric, and positive definite matrices: FEM and FD methods. Skyline solvers are usually based on Choleski factorization

Linear Systems of Equations
Special Matrices

Symmetric, Positive Definite Coefficient Matrix

No pivoting needed

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{L}}}\,\overline{\overline{\mathbf{U}}} = \overline{\overline{\mathbf{U}}}^{\dagger}\,\overline{\overline{\mathbf{U}}}$$

Choleski Factorization

$$\overline{\overline{\mathbf{U}}}^{\dagger} = [m_{ij}]$$

where

$$\left.\begin{array}{lcl} m_{kk} & = & \left(a_{kk} - \sum_{\ell=1}^{k-1} m_{k\ell}\overline{m}_{k\ell}\right)^{1/2} \\ m_{ik} & = & \frac{a_{ik} - \sum_{\ell=1}^{k-1} m_{i\ell}\overline{m}_{k\ell}}{m_{kk}}, \; i = k+1, \ldots n \end{array}\right\} k = 1, \ldots n$$