MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering
2.71/2.710 Optics
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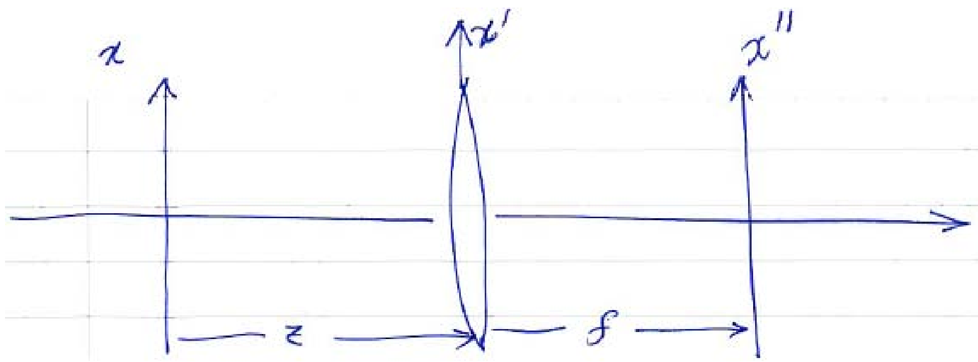
Supplement to Lecture 18
Derivation of Lens Fourier transforming property



$g(x)$ $\quad g_{L-}(x')$ $\quad g_{L+}(x')$ $\quad g_F(x'')$

$$g_{L-}(x',y') = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\iint g_{\text{in}}(x,y)e^{i\pi\frac{(x'-x)^2+(y'-y)^2}{\lambda z}}dxdy$$

$$g_{L+}(x',y') = g_{L-}(x',y')e^{-i\pi\frac{x'^2+y'^2}{\lambda f}}$$

$$g_F(x'',y'') = \frac{e^{i2\pi\frac{f}{\lambda}}}{i\lambda f}\iint g_{L+}(x',y')e^{i\pi\frac{(x''-x')^2+(y''-y')^2}{\lambda f}}dx'dy'$$

$$= \frac{e^{i2\pi\frac{z+f}{\lambda}}}{-\lambda^2 zf}\iint dx'dy'\iint dxdy\left(g(x,y)\times e^{i\pi\frac{x'^2+x^2-2xx'+y'^2+y^2-2yy'}{\lambda z}}\times\right.$$
$$\left.e^{-i\pi\frac{x'^2+y'^2}{\lambda f}}\times e^{i\pi\frac{x''^2+x'^2-2x''x'+y''^2+y'^2-2y''y'}{\lambda f}}\right)$$

$$= \frac{e^{i2\pi\frac{z+f}{\lambda}}}{-\lambda^2 zf}e^{i\pi\frac{x''^2+y''^2}{\lambda f}}\iint dxdy\; g(x,y)\times e^{i\pi\frac{x^2+y^2}{\lambda z}}\times C$$

$$C = \iint e^{i\pi\frac{x'^2+y'^2}{\lambda z}}e^{-i2\pi\left[\frac{x'}{\lambda}\left(\frac{x}{z}+\frac{x''}{f}\right)+\frac{y'}{\lambda}\left(\frac{y}{z}+\frac{y''}{f}\right)\right]}dx'dy'$$

From Goodman p. 14:

$$e^{i\pi(a^2x^2+b^2y^2)} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{i}{|ab|} e^{-i\pi\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)}, \quad a = b = (\sqrt{\lambda z})^{-1}$$

$$C = \mathcal{F}(e^{i\pi\frac{x'^2+y'^2}{\lambda z}})|_{u=\frac{1}{\lambda}(\frac{x}{z}+\frac{x''}{f}),\ v=\frac{1}{\lambda}(\frac{y}{z}+\frac{y''}{f})}$$

$$\begin{aligned}
\rightarrow C &= i\lambda z e^{-i\pi\lambda z\left[\frac{1}{\lambda^2}\left(\frac{x}{z}+\frac{x''}{f}\right)^2+\frac{1}{\lambda^2}\left(\frac{y}{z}+\frac{y''}{f}\right)^2\right]} \\
&= i\lambda z e^{-i\pi\frac{z}{\lambda}\left(\frac{x^2}{z^2}+\frac{x''^2}{f^2}+\frac{2xx''}{zf}+\frac{y^2}{z^2}+\frac{y''^2}{f^2}+\frac{2yy''}{zf}\right)} \\
&= i\lambda z e^{-i\pi\frac{x^2+y^2}{\lambda z}} e^{-i\pi\frac{x''^2+y''^2}{\lambda f^2}z} e^{-i2\pi\frac{xx''+yy''}{\lambda f}}
\end{aligned}$$

$$\begin{aligned}
g_F(x'', y'') = &\frac{e^{i2\pi\frac{z+f}{\lambda}}}{-\lambda^2 zf} e^{i\pi\frac{x''^2+y''^2}{\lambda f}} \iint dxdy\ g(x,y) e^{i\pi\frac{x^2+y^2}{\lambda z}} \times \\
&i\lambda z e^{-i\pi\frac{x^2+y^2}{\lambda z}} e^{-i\pi\frac{x''^2+y''^2}{\lambda f^2}z} e^{-i2\pi\frac{xx''+yy''}{\lambda f}} \\
&= \frac{e^{i2\pi\frac{z+f}{\lambda}}}{i\lambda f} e^{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)} \iint dxdy\ g(x,y) e^{-i2\pi\frac{xx''+yy''}{\lambda f}} \\
&= \frac{e^{i2\pi\frac{z+f}{\lambda}}}{i\lambda f} e^{i\pi\frac{x''^2+y''^2}{\lambda f}(1-\frac{z}{f})} G\left(\frac{x''}{\lambda f}, \frac{y''}{\lambda f}\right)
\end{aligned}$$

where we define the function G as:

$$G(u,v) = \iint g(x,y) e^{-i2\pi(ux+vy)} dxdy = \mathcal{F}(g(x,y))|_{(u,v)}$$

Special case: $z = f$

$$g_F(x'', y'') = \frac{e^{i2\pi\frac{2f}{\lambda}}}{i\lambda f} G\left(\frac{x''}{\lambda f}, \frac{y''}{\lambda f}\right)$$