DETECTING REGULI AND PROJECTION THEORY

We have one more theorem about the incidence theory of lines in $\mathbb{R}^3$.

Theorem 0.1. *If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with $\le B$ lines in any plane or regulus, and if $B \ge L^{1/2}$, then the number of intersection points of $\mathfrak{L}$ is $\lesssim BL$.*

This theorem is an improvement of our earlier estimate on 3-rich points.

Theorem 0.2. *If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with $\le B$ lines in any plane, and if $B \ge L^{1/2}$, then $|P_3(\mathfrak{L})| \lesssim BL$.*

Recall that $P_k(\mathfrak{L})$ is the set of k-rich points of $\mathfrak{L}$ – in particular $P_2(\mathfrak{L})$ is the set of intersection points of $\mathfrak{L}$.

The proof of Theorem 0.2 uses the theory of critical points and flat points. We can't directly apply this theory to Theorem 0.1, because a point lying in two lines in a surface may be neither critical nor flat. So we will have to modify/refine these tools.

Let's package what we need about critical/flat points into one lemma for generalization.

Plane detection lemma. *For any polynomial P in $\mathbb{R}[x_1, x_2, x_3]$, we can associate a list of polynomials SP with the following properties.*

(1) *$Deg SP \le 3 Deg P$.*
(2) *If x is contained in three lines in $Z(P)$, then $SP(x) = 0$.*
(3) *If P is irreducible and SP vanishes on $Z(P)$, then $Z(P)$ is a plane.*

Roughly speaking, SP has the job of detecting whether $Z(P)$ looks like a plane. If $SP(x) = 0$, then it (roughly) means that $Z(P)$ looks kind of like a plane near x. If SP vanishes on $Z(P)$ (and P irreducible), then it means that $Z(P)$ is a plane. We will refine this technique and build a polynomial RP that detects whether $Z(P)$ looks like a regulus.

Regulus detection lemma. *For any polynomial P in $\mathbb{R}[x_1, x_2, x_3]$, we can associate a list of polynomials RP with the following properties.*

(1) *$Deg RP \le C Deg P$.*
(2) *If x is contained in two lines in $Z(P)$, then $RP(x) = 0$.*
(3) *If P is irreducible and RP vanishes on $Z(P)$, and if there is a non-special point x contained in two lines in $Z(P)$, then $Z(P)$ is a regulus.*

The proof of Theorem 0.1 is essentially the proof of Theorem 0.2 using the regulus detection lemma instead of the plane detection lemma. We will include the details later, but there are no significant new ingredients. The new tool is the regulus detection lemma.

(The two detection lemmas are quite similar. The regulus detection lemma has an extra condition in the last item: "if there is a non-special point x contained in two lines in $Z(P)$". Recall that x is special if it is either critical or flat. This condition is not very elegant, but it will be easy to meet in the application to Theorem 0.1. If all the intersection points were critical or flat, then we could handle the situation with the plane detection lemma anyway.)

The regulus detection lemma is based on ideas about "ruled surfaces" developed by Salmon and Cayley in the 19th century. They proved the first interesting example of a detection lemma.

1. Ruled surfaces and flecndes

We consider algebraic surfaces in $\mathbb{C}^3$ in this section.

Suppose P is an irreducible polynomial. How many lines can there be in $Z(P)$? There can be infinitely many, which happens for planes, reguli, cones, and cylinders. There are actually many other examples.

For instance, consider a polynomial map $\Phi : \mathbb{C}^2 \to \mathbb{C}^3$ of the form $\Phi(s,t) = \Phi_1(s)t + \Phi_0(s)$. The image contains infinitely many lines (fix s and let t vary). Also, the image is contained in $Z(P)$ for some P. (Is the image exactly $Z(P)$ for some P?)

An algebraic surface $Z(P)$ is called 'ruled' if each $x \in Z(P)$ lies in a line $\subset Z(P)$. Now we can ask a more refined question. If P is irreducible of a given degree, and $Z(P)$ is not ruled, then how many lines can there be in $Z(P)$?

Theorem 1.1. *If P is an irreducible polynomial in $\mathbb{C}[z_1, z_2, z_3]$, then either $Z(P)$ is ruled or the number of lines in $Z(P)$ is $\leq C(degP)^2$.*

This theorem follows from the work of Salmon and Cayley from the 1800's. It appears in Salmon's book *A Treatise on the Analytic Geometry of Three Dimensions.* Chapter XIII deals with ruled surfaces, and ... First published in ?

In particular, the theorem follows from Salmon and Cayley's work on the flecnode polynomial. They proved the following result.

Ruled surface detection lemma. *For any polynomial P in $\mathbb{C}[x_1, x_2, x_3]$, we can define a finite set of polynomials FP with the following properties.*

(1) $DegFP \leq CDegP$.
(2) If x is contained in a line in $Z(P)$, then $FP(x) = 0$.
(3) If FP vanishes on $Z(P)$, then $Z(P)$ is ruled.

The polynomial FP is called the flecnode polynomial. In fact FP is a single polynomial (not a set of several polynomials), but this fact doesn't matter that much in applications, and it's easier to prove the Ruled surface detection lemma in the form above. Given the flecnode polynomial, the estimate on the number of lines in a non-ruled surface follows from Bezout's theorem.

Salmon defined FP (and gave a formula for it), and he proved properties 1 and 2. Then Cayley proved property 3. (See pages 277-78 of Salmon's book.)

We will try to explain the main ideas in this type of detection lemmas. We will try to give a fairly general point of view about how to prove this type of lemma, and we will try to avoid writing long formulas. We will give a complete proof of the regulus detection lemma, and we will give the main ideas of the proof of the ruled surface detection lemma.

We say a point $z \in \mathbb{C}^3$ is flecnodal (for P) if there exists a non-zero vector V so that P vanishes in the direction V to fourth order. We write $\nabla_V^s P$ to denote the s^{th} directional derivative of P in the direction V. We say P is flecnodal at z if there exists a non-zero vector V so that

$$0 = \nabla_V P(z) = \nabla_V^2 P(z) = \nabla_V^3 P(z). \tag{1}$$

If z is contained in a line in $Z(P)$, and if V is tangent to the line, then equation (1) holds.

It can be helpful to expand this expression in terms of derivatives of P in the coordinate directions. V is a vector $(V_1, V_2, V_3) \in \mathbb{C}^3$. For a multi-index $I = (i_1, i_2, i_3)$, we write V^I for $V_1^{i_1} ... V_3^{i_3}$, ∂_I for $\frac{\partial^{i_1}}{\partial z_1^{i_1}} ... \frac{\partial^{i_3}}{\partial z_3^{i_3}}$, and $I!$ for $i_1! \cdot i_2! \cdot i_3!$.

$$\nabla_V^s P(z) := \sum_{|I|=s} I! V^I \partial_I P(z).$$

Salmon defined his polynomial FP and proved that $FP(z) = 0$ if and only if z is a flecnodal point. So the flecnode polynomial detects flecnodal points. These facts boil down to the following lemma.

Lemma 1.2. *Consider the set of equations*

$$0 = \sum_{|I|=s} V^I a_I, s = 1, 2, 3. \tag{2}$$

In these equations, a_I are parameters in $\mathbb{C}$. We let a be the vector with components a_I, so $a \in \mathbb{C}^M$ for some M.

$$Sol := \{a \in \mathbb{C}^M | \text{ Equation (2) has a non-zero solution } V \in \mathbb{C}^3\}.$$

The set Sol is an algebraic set in $\mathbb{C}^M$. In other words, Sol is the zero set of some list of polynomials G.

Given the lemma, we define $FP(z) = G(\partial_I P(z))$. If G is a set of polynomials of degree $\leq C$ in a_I, then FP is a set of polynomials of degree $\leq C(degP)$ in z. By the lemma, a point z is flecnodal if and only if $FP(z) = 0$.

In summary, given the Lemma 1.2, we can immediately define FP and prove properties 1 and 2 of the ruled surface detection lemma.

Lemma 1.2 is part of an area called projection theory. It's a special case of the fundamental theorem of projection theory. We introduce projection theory and prove the fundamental theorem in the next section.

2. Projection Theory

Let $\mathbb{F}$ be a field. Recall that an algebraic set in $\mathbb{F}^M$ is just the zero set of a finite list of polynomials. Suppose that Z is an algebraic set in $\mathbb{F}^m \times \mathbb{F}^n$, and we consider the projection of Z onto the second factor. Is the projection also an algebraic set?

In general the answer is no. Let's consider two examples. We begin working over the field $\mathbb{R}$ where everything is as simple as possible to visualize.

Example 2.1. *(Circle example) Let Z be the zero set of $x^2 + y^2 - 1$ in $\mathbb{R}^2$. If we project Z to the x axis we get the closed segment $[-1, 1]$. This is not an algebraic set.*

Example 2.2. *(Hyperbola example) Let Z be the zero set of $xy = 1$ in $\mathbb{R}^2$. If we project Z to the x-axis, we get $\mathbb{R} \setminus \{0\}$. This is not an algebraic set.*

Projection theory studies this situation. What kind of structure do the projections have? Are there some situations where the projection is an algebraic set?

What would happen if we work over $\mathbb{C}$ instead of $\mathbb{R}$? The example with the circle gets better. If we let Z be the zero set of $x^2 + y^2 - 1$ in $\mathbb{C}^2$, then the projection of Z to the x axis is $\mathbb{C}$. But the hyperbola example is the same as before – if we work over $\mathbb{C}$, the image of the projection is $\mathbb{C} \setminus \{0\}$.

We can loosely describe the situation with the hyperbola in the following way. For each $x \in \mathbb{C} \setminus \{0\}$, there is a unique solution to the equation $xy - 1 = 0$. As x approaches zero, this solution $y(x)$ tends to infinity. In some sense, when x is equal to zero, the solution is "at infinity". We can make this precise by working with projective space. Instead of $y \in \mathbb{F}^n$, we can consider $y \in \mathbb{FP}^n$. Instead of starting with an algebraic set $Z \subset \mathbb{F}^m \times \mathbb{F}^n$, we can start with an algebraic set $Z \subset \mathbb{F}^m \times \mathbb{FP}^n$. If $\mathbb{F}$ is algebraically closed and if we work projectively, then the projection of Z is also algebraic.

Working with y in projective space is equivalent to using polynomials that are homogeneous in y. We can phrase the fundamental theorem of projection theory in the following way.

Fundamental Theorem of Projection Theory. *Suppose that $Q(x, y)$ is a finite list of polynomials in $x \in \mathbb{F}^m$ and $y \in \mathbb{F}^n$, each of which is homogeneous in y. Let $SOL \subset \mathbb{F}^m$ be the set*

$$SOL := \{x \in \mathbb{F}^m | \text{the equation } Q(x, y) = 0 \text{ has a non-zero solution } y \in \mathbb{F}^n\}.$$

If $\mathbb{F}$ is algebraically closed, then SOL is an algebraic set.

For example, consider the equations in Lemma 1.2. We have the equations $0 = \sum_{|I|=s} a_I V^I$, for $s = 1, 2, 3$. Each equation is homogeneous in V. By the fundamental theorem of projection theory, the set of a so that these equations have a non-zero solution $V \in \mathbb{C}^3 \setminus \{0\}$ is an algebraic set. So Lemma 1.2 is a corollary of the fundamental theorem of projection theory.

3. Proof of the fundamental theorem of projection theory

Let $\mathbb{F}$ be any field. Let $Q_j(x, y)$ be homogeneous in y with degree d_j. If we think of x as a parameter, for each x, we get $Q_{j,x}(y)$, a polynomial in y which is homogeneous of degree d_j. We let $I(x) \subset \mathbb{F}[y]$ be the ideal spanned by the polynomials $Q_{j,x}(y)$.

This ideal is homogeneous. Recall that for any polynomial Q we write $Q_{=d}$ for the degree d part of Q. An ideal is homogeneous if for any $Q \in I$, and any d, we have $Q_{=d} \in I$ also. In particular, any ideal generated by homogeneous polynomials is homogeneous. We let $I(x)_{=d}$ be the homogeneous degree d polynomials in $I(x)$.

Proposition 3.1. *For any integers $d, B \geq 0$, the set $\{x \in \mathbb{F}^m | dim I(x)_{=d} \leq B\}$ is an algebraic set.*

This proposition follows from the homogeneity of $Q(x, y)$ (in y).

Let $H_{=d} \subset \mathbb{F}[y_1, ..., y_n]$ be the degree d homogeneous polynomials.

Proof. Consider the multiplication map $M(x)_{=d} : \oplus_j H_{=d-d_j} \to I(x)_{=d}$, given by

$$M_{=d}(R) := \sum_j Q_{j,x} R_j.$$

Since R_j is homogeneous of degree $d - d_j$ and $Q_{j,x}$ is homogeneous of degree d_j, we see that $M_{=d}(R)$ is homogeneous of degree d. Since $I(x)$ is the ideal spanned by $Q_{j,x}$, the image of $M(x)_{=d}$ is in $I(x)$. So we see that $M(x)_{=d}$ is a linear map to $I(x)_{=d}$ as claimed.

The key point of the proof is that $M(x)_{=d}$ is surjective! This follows from the homogeneity. Suppose that $f \in I(x)_{=d}$. By definition, f is degree d and $f = \sum_j Q_{j,x} f_j$ for some polynomials f_j. But since $Q_{j,x}$ is homogeneous of degree d_j, we see that $f = \sum_j Q_{j,x} f_{j,=d-d_j}$. So f is in the image of $M(x)_{=d}$.

The linear map $M(x)_{=d}$ can be described by a matrix. The dimension of $I(x)_{=d}$ is exactly the rank of this matrix. The entries of the matrix are polynomials in x. The matrix $M(x)_{=d}$ has rank $\leq B$ if and only if each $(B+1) \times (B+1)$ subdeterminant vanishes. Therefore, the set of matrices $M(x)_{=d}$ with rank $\leq B$ is an algebraic set. □

Proposition 3.2. *For any integers $d, B \geq 0$, the set $\{x \in \mathbb{F}^m | \mathbb{F}[y]/I(x)$ is infinite dimensional$\}$ is an algebraic set.*

Proof. The first step is to see that $\mathbb{F}[y]/I(x)$ is infinite dimensional if and only if $I(x)_{=d}$ is a proper subspace of $H_{=d}$ for every $d \geq 0$. Indeed, if $I(x)_{=d} = H_{=d}$ for some d, then $I(x)$ contains all homogeneous polynomials of degree $\geq d$, and so $\mathbb{F}[y]/I(x)$ is finite dimensional. The other direction is straightforward.

So the set of x where $\mathbb{F}[y]/I(x)$ is infinite dimensional is exactly

$$\bigcap_{d \geq 0} \{x \in \mathbb{F}^m | dim I(x)_{=d} \leq dim H_{=d} - 1\}.$$

By the last proposition this is a countable intersection of algebraic sets. By the Noetherian property of $\mathbb{F}[y]$, the intersection stabilizes after finitely many values of d, and so the infinite intersection is also an algebraic set. □

Proposition 3.3. *If $\mathbb{F}$ is algebraically closed and $I \subset \mathbb{F}[y]$ is a homogeneous ideal, then $Z(I)$ contains a non-zero point if and only if $\mathbb{F}[y]/I$ is infinite dimensional (as a vector space over $\mathbb{F}$).*

Proof. We begin with the easy direction. Suppose that $0 \neq y$ lies in $Z(I)$. By homogeneity, the line through 0 and y also lies in $Z(I)$. Now we consider the evaluation map from $\mathbb{F}[y]/I$ to the functions on this line. Since $\mathbb{F}$ is algebraically closed, there are infinitely many points on the line. For any finite subset of the points on the line, a polynomial can take arbitrary values. Therefore, the rank of the evaluation map is infinite, and the dimension of $\mathbb{F}[y]/I$ is infinite.

Suppose instead that 0 is the only point in $Z(I)$. By the Nullstellensatz, the radical of I is the ideal generated by $y_1, ..., y_n$. This use of the Nullstellensatz uses the fact that $\mathbb{F}$ is algebraically closed. If I happens to be radical, then $\mathbb{F}[y]/I$ is $\mathbb{F}$, and we are done. In not, then we get some finite sequence of ideals $I = I_0 \subset I_1 \subset ... \subset I_J = (y_1, ..., y_n)$, where each ideal is formed by adding a radical element to the previous ideal. By backwards induction on j, we check that $R_j = \mathbb{F}[y]/I_j$ is finite dimensional. This is true for $j = J$. Now R_{j-1} is formed by adjoining a nilpotent element to R_j. The inductive step is then straightforward. □

Assembling these three propositions gives the fundamental theorem of projection theory.

4. Taking stock

We have now defined the polynomial FP. We proved that $degFP \leq \alpha degP$ for some constant α. We proved that $FP(x) = 0$ if and only if the point x is flecnodal. If x lies in a line in $Z(P)$, then x is obviously flecnodal and so $FP(x) = 0$.

Suppose that P is irreducible and that $Z(P)$ contains $> \alpha(degP)^2$ lines. The polynomial FP vanishes on each of these lines. Since the number of lines is $> (degP)(degFP)$, it follows that P divides FP, and so $FP = 0$ on $Z(P)$. We conclude that every point of $Z(P)$ is flecnodal: at every point there is a direction V in which P vanishes to fourth order.

The next step is to prove that the surface is actually ruled. Because every point is flecnodal, the surface "looks nearly ruled" at every point. The next step is a local-to-global argument: because there is locally always a line nearly in the surface, the surface is globally ruled. This argument is quite different - it has to do more with differential geometry than with algebra. We discuss it more next time.

Finally, we note that our set up so far is pretty flexible. For example, suppose we define a point z to be t-flecnodal if there is a non-zero vector V so that $\nabla_v^s P(x) = 0$ for all s from 1 to t. By the same argument as above, we can construct a finite set of polynomials $F_t P$ with degree $\leq \alpha(t) degP$ so that z is t-flecnodal if and only if $F_t P(z) = 0$. If P is irreducible and $Z(P)$ contains $> \alpha(t)(degP)^2$ lines, then every point of $Z(P)$ is t-flecnodal. The flecnode is defined with $t = 3$, because that's the smallest value of t where the local-to-global argument works. But we can choose to work with any value of t, and it's actually a little easier to prove the local-to-global result with $t = 4$ or $t = 10$...

If a point lies in two lines in $Z(P)$ we can find two linearly independent vectors V_1, V_2 where $\nabla_V^s P(z) = 0$ for all s. With a little modification of the technology, we can build a polynomial RP that vanishes whenever there are two independent directions in which P vanishes to order 4. We pick that up next time.