4 Lecture 4 (Notes: J. Pascaleff)

4.1 Geometry of $V \oplus V^*$

Let V be an n-dimensional real vector space, and consider the direct sum $V \oplus V^*$. This space has a natural symmetric bilinear form, given by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$$

for $X, Y \in V$, $\xi, \eta \in V^*$. Note that the subspaces V and V^* are null under this pairing.

Choose a basis $e_1, e_2, \dots, e_n$ of V, and let $e^1, e^2, \dots, e^n$ be the dual basis for V^*. Then the collection

$$e_1+e^1, e_2+e^2, \dots, e_n+e^n, \quad e_1-e^1, e_2-e^2, \dots, e_n-e^n$$

is a basis for $V \oplus V^*$, and we have

$$\langle e_i+e^i, e_i+e^i\rangle = 1$$
$$\langle e_i-e^i, e_i-e^i\rangle = -1,$$

whereas for $i \neq j$,

$$\langle e_i \pm e^i, e_j \pm e^j\rangle = 0$$

Thus the pairing $\langle\cdot,\cdot\rangle$ is non-degenerate with signature (n,n), a so-called "split signature." The symmetry group of the structure consisting of $V \oplus V^*$ with the pairing $\langle\cdot,\cdot\rangle$ is therefore

$$\mathrm{O}(V \oplus V^*) = \{A \in \mathrm{GL}(V \oplus V^*) : \langle A\cdot, A\cdot\rangle = \langle\cdot,\cdot\rangle\} \cong \mathrm{O}(n,n).$$

Note that $\mathrm{O}(n,n)$ is not a compact group.

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We have a natural orientation on $V \oplus V^*$ coming from the canonical isomorphisms

$$\det\,(V \oplus V^*) = \det\, V \otimes \det\, V^* = \mathbf{R}.$$

The symmetry group of $V \oplus V^*$ therefore naturally reduces to $\mathrm{SO}(n,n)$.

The Lie algebra of $\mathrm{SO}(\mathrm{V} \oplus \mathrm{V}^*)$ is

$$\mathfrak{so}(V \oplus V^*) = \{Q : \langle Q\cdot, \cdot\rangle + \langle \cdot, Q\cdot\rangle\}.$$

By way of the non-degenerate bilinear form on $V \oplus V^*$, we may identify $V \oplus V^*$ with its dual, and so we may write

$$\mathfrak{so}(V \oplus V^*) = \{Q : Q + Q^* = 0\}.$$

We may decompose $Q \in \mathfrak{so}(V \oplus V^*)$ in view of the splitting $V \oplus V^*$:

$$Q = \begin{pmatrix} A & \beta \\ B & D \end{pmatrix},$$

where

$$\begin{array}{cc} A : V \to V & \beta : V^* \to V \\ B : V \to V^* & D : V^* \to V^* \end{array}$$

The condition that $Q + Q^* = 0$ means now

$$Q^* = \begin{pmatrix} D^* & \beta^* \\ B^* & A^* \end{pmatrix} = -Q,$$

or $D^* = -A$, $\beta^* = -\beta$, and $B^* = -B$. The necessary and sufficient conditions that A, β, B, D give an element of $\mathfrak{so}(V \oplus V^*)$ are therefore

$$A \in \mathrm{End}V, \quad \beta \in \wedge^2 V, \quad B \in \wedge^2 V^*, \quad D = -A^*.$$

Thus we may identify $\mathfrak{so}(V \oplus V^*)$ with

$$\mathrm{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^*.$$

This decomposition is consistent with the fact that, for any vector space E with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, we have

$$\mathfrak{so}(E) = \wedge^2 E.$$

In the case of $E = V \oplus V^*$ this gives

$$\mathfrak{so}(V \oplus V^*) = \wedge^2(V \oplus V^*) = \wedge^2 V \oplus (V \otimes V^*) \oplus \wedge^2 V^*,$$

and the term $V \otimes V^*$ is just $\mathrm{End}(V)$.

Of particular note is the fact that the "usual" symmetries $\mathrm{End}(V)$ of V are contained in the symmetries of $V \oplus V^*$. (Since V is merely a vector space with no additional structure, its symmetry group is $\mathrm{GL}(V)$, with Lie algebra $\mathfrak{gl}(V) = \mathrm{End}(V)$.)

Now we examine how the different parts of the decomposition

$$\mathfrak{so}(V \oplus V^*) = \mathrm{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^*$$

act on $V \oplus V^*$.

Any $A \in \mathrm{End}(V)$ corresponds to the element

$$Q_A = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \in \mathfrak{so}(V \oplus V^*).$$

Which acts on $V \oplus V^*$ as the linear transformation

$$e^{Q_A} = \begin{pmatrix} e^A & 0 \\ 0 & ((e^A)^*)^{-1} \end{pmatrix} \in \mathrm{SO}(V \oplus V^*)$$

Since any transformation $T \in \mathrm{GL}^+(V)$ of positive determinant is e^A for some $A \in \mathrm{End}(V)$. We can regard $\mathrm{GL}^+(V)$ as a subgroup of $\mathrm{SO}(V \oplus V^*)$. In fact the map

$$P \mapsto \begin{pmatrix} P & 0 \\ 0 & (P^*)^{-1} \end{pmatrix}$$

gives an injection of $\mathrm{GL}(V)$ into $\mathrm{SO}(V \oplus V^*)$.

Thus, once again, the usual symmetries $\mathrm{GL}(V)$ may be regarded as part of a larger group of symmetries, namely $\mathrm{SO}(V \oplus V^*)$. This is the direct analog of the same fact at the level of Lie algebras.

Now consider a 2-form $B \in \wedge^2 V^*$. This element corresponds to

$$Q_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \in \mathfrak{so}(V \oplus V^*),$$

which acts $V \oplus V^*$ as the linear transformation

$$e^B = e^{Q_B} = \exp \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + 0 = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

since $Q_B^2 = 0$. More explicitly, e^Q_B is the map

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X \\ \xi + B(X) \end{pmatrix} = \begin{pmatrix} X \\ \xi + i_X B \end{pmatrix}.$$

Thus B gives rise to a shear transformation which preserves the projection onto V. These transformations are called *B-fields.*

The case of a bivector $\beta \in \wedge^2 V$ is analogous to that of a 2-form: β corresponds to

$$Q_\beta = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

which acts on $V \oplus V^*$ as

$$e^\beta = e^{Q_\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} X \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} X + i_\xi \beta \\ \xi \end{pmatrix},$$

or in other words a shear transformation preserving projection onto V^*. These are called *β-field* transformations.

In summary, the natural structure of $V \oplus V^*$ is such that we may regard three classes of objects defined on V, namely, endomorphisms, 2-forms, and bivectors, as orthogonal symmetries of $V \oplus V^*$.

4.2 Linear Dirac structures

A subspace $L \subset V \oplus V^*$ is called *isotropic* if

$$\langle x, y \rangle = 0 \quad \text{for all } x, y \in L.$$

If V has dimension n, then the maximal dimension of an isotropic subspace in $V \oplus V^*$ is n. Isotropic subspaces of the maximal dimension are called *linear Dirac structures* on V.

Examples of linear Dirac structures on V are

1. V
2. V^*.
3. $e^B V = \{X + i_X B : X \in V\}$, which is simply the graph Γ_B of the map $B : V \to V^*$.
4. $e^\beta \cdot V^* = \{i_\xi \beta + \xi : \xi \in V^*\}$.
5. In general, $A \cdot V$, where $A \in \mathrm{O}(V \oplus V^*)$.

Exercise. If D is a linear Dirac structure on V, such that the projection onto to V, $\pi_V(D) = V$, then there is a unique $B : V \to V^*$ such that $D = e^B V$. Specifically $B = \pi_{V^*} \circ (\pi_V|D)^{-1}$.

A further example of a linear Dirac structure is given as follows: let $\Delta \subset V$ be any subspace of dimension d. Then the annihilator of Δ, $\mathrm{Ann}(\Delta)$, consisting of all 1-forms which vanish on Δ is a subspace of V^* of dimension $n - d$. The space

$$D = \Delta \oplus \mathrm{Ann}(\Delta) \subset V \oplus V^*$$

is then isotropic of dimension n, and is hence a linear Dirac structure.

When we apply a B-field to a Dirac structure of this kind, we get

$$\begin{aligned} e^B(\Delta \oplus \mathrm{Ann}(\Delta)) &= \{X + \xi + i_X B : X \in \Delta, \xi \in \mathrm{Ann}(\Delta)\} \\ &= e^B(\Delta) \oplus \mathrm{Ann}(\Delta). \end{aligned}$$

We define the *type* of a Dirac structure D to be $\mathrm{codim}(\pi_V(D))$. The computation above shows that a B-field transformation cannot change the type of a Dirac structure.

What matters in this computation is not so much B itself as it is the pullback f^*B of B under the inclusion $f : \Delta \to V$. Indeed, if $f^*B = f^*B'$, then

$$0 = i_X(f^*B - f^*B') = f^*(i_X B - i_X B').$$

This means that $i_X B - i_X B' \in \mathrm{Ann}(\Delta)$, and so

$$e^B(\Delta) \oplus \mathrm{Ann}(\Delta) = e^{B'}(\Delta) \oplus \mathrm{Ann}(\Delta).$$

Generalizing this observation, let $f : E \to V$ be the inclusion of a subspace E of V, and let $\epsilon \in \wedge^2 E^*$. Then define

$$L(E, \epsilon) = \{X + \xi \in E \oplus V^* : f^*\xi = i_X \epsilon\},$$

which is a linear Dirac structure. Note that when $\epsilon = 0$,

$$L(E, 0) = E \oplus \mathrm{Ann}(E).$$

Otherwise, $L(E, \epsilon)$ is a general Dirac structure.

Conversely, the subspace E and 2-form ϵ may be reconstructed from a given Dirac structure L. Set

$$E = \pi_V(L) \subset V.$$

Then

$$\begin{aligned} L \cap V^* &= \{\xi : \langle \xi, L \rangle = 0\} \\ &= \{\xi : \xi(\pi_V(L)) = 0\} \\ &= \mathrm{Ann}(E). \end{aligned}$$

We can define a map from E to $V^*/L \cap V^*$ by taking $e \in E$ first to $(\pi_V|L)^{-1}(e) \in L$, and then projecting onto $V^*/L \cap V^*$; this yields

$$\epsilon : E \to V^*/L \cap V^* = V^*/\mathrm{Ann}(E) = E^*.$$

Then we have $\epsilon \in \wedge^2 E^*$, and $L = L(E, \epsilon)$.

In an analogous way, we could consider Dirac structures $L = L(F, \gamma)$, where $F = \pi_{V^*}(L)$, and $\gamma : F \to F^*$.

Exercise. Let $\mathrm{Dir}_k(V)$ be the space of Dirac structures of type k. Determine dim $\mathrm{Dir}_k(V)$. Compare this to the usual stratification of the Grassmannian $\mathrm{Gr}_k(V)$.

A B-field transformation cannot change the type of a Dirac structure, since

$$e^B L(E, \epsilon) = L(E, \epsilon + f^* B).$$

However, a β-field transform can. Express a given Dirac structure L as $L(F, \gamma)$, with $g : F \to V^*$ an inclusion, and $\gamma \in \wedge^2 F^*$. Let $E = \pi_V(L)$, which contains $L \cap V = \mathrm{Ann}(F)$. Thus

$$E/L \cap V = E/\mathrm{Ann}(F) = \mathrm{Im}\ \gamma,$$

and so

$$\dim E = \dim L \cap V + \mathrm{rank}\ \gamma.$$

Since rank γ is always even, if we change γ to $\gamma + g^*\beta$, we can change dim E by an even amount.

The space $\mathrm{Dir}(V)$ of Dirac structures has two connected components, one consisting of those of even type, and one consisting of those of odd type.

4.3 Generalized metrics

There is another way to see the structure of $\mathrm{Dir}(V)$. Let $C_+ \subset V \oplus V^*$ be a maximal subspace on which the pairing $\langle \cdot, \cdot \rangle$ is positive definite, e.g., the space spanned by $e_i + e^i$, $i = 1, \ldots, n$. Let $C_- = C_+^\perp$ be the orthogonal complement. Then $\langle \cdot, \cdot \rangle$ is negative definite on C_-.

If L is a linear Dirac structure, then $L \cap C_\pm = \{0\}$, since L is isotropic. Thus L defines an isomorphism.

$$L : C_+ \to C_-$$

such that $-\langle Lx, Ly \rangle = \langle x, y \rangle$, since $\langle x + Lx, y + Ly \rangle = 0$. By choosing isomorphism between $C_\pm$ and $\mathbf{R}^n$ with the standard inner product, any $L \in \mathrm{Dir}(V)$ may be regarded as an orthogonal transformation of $\mathbf{R}^n$, and conversely. Thus $\mathrm{Dir}(V)$ is isomorphic to $\mathrm{O}(n)$ as a space. The two connected components of $\mathrm{O}(n)$ correspond in some way to the two components of $\mathrm{Dir}(V)$ consisting of Dirac structures of even and odd type.

Observe that because C_+ is transverse to V and V^*, the choice of C_+ is equivalent to the choice of a map $\gamma : V \to V^*$ such that the graph Γ_γ is a positive definite subspace, i.e., for $0 \neq x \in V$,

$$\langle x + \gamma(x), x + \gamma(x) \rangle = \gamma(x, x) > 0.$$

Thus if we decompose γ into $g + b$, where g is the symmetric and b the anitsymmetric part, then g must be a positive definite metric on V. The form $g + b$ is called a *generalized metric* on V. A generalized metric defines a positive definite metric on $V \oplus V^*$, given by

$$\langle \cdot, \cdot \rangle|_{C_+} - \langle \cdot, \cdot \rangle|_{C_-}$$

Exercise. Given $A \in \mathrm{O}(n)$, determine explicitly the Dirac structure L_A determined by the map $\mathrm{O}(n) \to \mathrm{Dir}(V)$.