HOPF FIBRATIONS, WHITEHEAD THEOREM

Example 0.1. Applying the LES of a fibration to the Hopf fibration

$$S^1 \to S^3 \to S^2$$

we deduce

$$\pi_2(S^2) \cong \mathbb{Z}$$
$$\pi_k(S^3) \cong \pi_k(S^2) \quad \text{for } k > 2$$

In particular, assuming the result $\pi_3(S^3) = \mathbb{Z}$, we may deduce that $\pi_3(S^2) = \mathbb{Z}$, generated by the Hopf map η.

1. Hopf invariant

Suppose that $f : S^{2n-1} \to S^n$ is a based map. The cofiber $C(f)$ has cohomology

$$\widetilde{H}^*(C(f)) = \mathbb{Z}\{x_n\} \oplus \mathbb{Z}\{x_{2n}\}$$

generated by x_n in degree n and x_{2n} in degree $2n$. The Hopf invariant of f is defined by the formula

$$(x_n)^2 = HI(f) \cdot x_{2n}.$$

Because there are two choices of generators of each of these groups, the Hopf invariant is determined only up to sign.

On the homework, you showed that for the Hopf fibration

$$S^1 \to S^3 \xrightarrow{\eta} S^2$$

the cofiber was given by

$$C(\eta) = \mathbb{C}P^2.$$

Replacing $\mathbb{C}$ with the real division algebras $\mathbb{H}$ and $\mathbb{O}$ gives similar fibrations

$$S^3 \to S^7 \xrightarrow{\nu} S^4$$
$$S^7 \to S^{15} \xrightarrow{\sigma} S^8$$

and the maps ν and σ also may be shown to have Hopf invariant 1.

It turns out that any real division algebra $\mathbb{D}$ gives rise to a "Hopf fibration". The following theorem of Adams therefore implies that the only (potentially non-associative) real division algebras are $\mathbb{R}$, $\mathbb{H}$, $\mathbb{C}$, and $\mathbb{O}$.

Theorem 1.1 (Adams)**.** The only maps of Hopf invariant 1 are η, ν, and σ.

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2. Whitehead theorem

Definition 2.1. A map $f : X \to Y$ is an n-equivalence if for all $x \in X$ the induced maps

$$f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$$

are isomorphisms for $k < n$ and is an epimorphism for $k = n$.

Remark 2.2. A space Z is n-connected if $\pi_k(Z, z) = 0$ for all $k \leq n$. The LES of a fibration implies that a map f is an n-equivalence if and only if the homotopy fiber $F(f)$ is $(n-1)$-connected.

We aim to prove the following theorem

Theorem 2.3. Suppose that Z is a CW-complex of dimension $< n$, and that $f : X \to Y$ is an n-equivalence. Then the induced map

$$[Z, X] \to [Z, Y]$$

is an isomorphism.

This theorem is true for $n = \infty$ as well. Applying it to $Z = Y$, we get

Corollary 2.4. If $f : X \to Y$ is a weak equivalence between CW-complexes, then it is a homotopy equivalence.

We will prove Theorem 2.3 in an intuitive way for $n < \infty$. The idea is that CW-complexes Z are built out of spheres, so homotopy groups are sufficient to detect isomorphisms on $[Z, -]$.

We shift to the pointed context with the following lemma.

Lemma 2.5. Suppose that X is pointed, and Z is well pointed. Then there is a split short exact sequence of pointed sets

$$[Z, X]_* \to [Z, X] \to \pi_0(X)$$

Proof. Since Z is well-pointed, there is a cofibration sequence

$$S^0 \to Z_+ \to Z = Z_+/S^0.$$

Now apply $[-, X]_*$, using the fact that the maps in the sequence above admit retractive splittings. □

It therefore suffices to prove

Theorem 2.6. Suppose that Z is a connected pointed CW-complex of dimension $< n < \infty$, and that $f : X \to Y$ is an n-equivalence of pointed spaces. Then the induced map

$$[Z, X]_* \to [Z, Y]_*$$

is an isomorphism.

Proof. By adding whiskers to the cells to make the attaching maps pointed, connected, pointed CW-complexes Z may be viewed as inductive cofiber sequences

$$\bigvee S^n \to Z^{[k]} \to Z^{[k+1]}.$$

where $Z^{[k]}$ is the k-skeleton. Applying $[-, X]_* \to [-, Y]_*$ to the above sequence, induction on k and the 5-lemma gives the result. □

Remark 2.7. The application of the 5-lemma in the proof above is delicate, because in the sequence

$$[\Sigma Z^{[k]}, X]_* \to [\bigvee S^{k+1}, X]_* \to [Z^{[k+1]}, X]_* \to [Z^{[k]}, X]_* \to [\bigvee S^k, X]_*$$

the last three terms are potentially only pointed sets, not groups. This is OK: the five lemma works in this context, provided the second term acts on the third term, with orbits the preimages of elements of the fourth term. This action is given by dualizing a generalization of the pinch map: for a map $g : A \to B$ there is a coaction

$$C(g) \to \Sigma A \vee C(g).$$

Remark 2.8. The case of $n = \infty$ is not obviously implied by the previous theorem, because it is not generally true that the canonical map

$$[Z, X]_* \to \varprojlim [Z^{[n]}, X]_*$$

is an isomorphism.