18.905 Problem Set 11

Due Wednesday, November 29 (post-break) in class

Five questions. Do all five.

1. Hatcher, exercise 2 on page 257.

2. Hatcher, exercise 7 on page 258.

3. Show that the exterior cup product $\overline{\smile}$ respects boundary homomorphisms. In other words, suppose $A \subset X$ and Y are spaces, R is a ring, $\alpha \in H^p(A; R)$, and $\beta \in H^q(Y; R)$. There are coboundary maps

   $$\delta_1 : H^p(A; R) \to H^{p+1}(X, A; R)$$
   $$\delta_2 : H^{p+q}(A \times Y; R) \to H^{p+q+1}(X \times Y, A \times Y; R)$$

   Show that $(\delta_1 \alpha) \overline{\smile} \beta = \delta_2(\alpha \overline{\smile} \beta)$.

4. A *generalized cohomology theory* is to a generalized homology theory as cohomology is to homology; i.e., it is a collection of contravariant functors E^n satisfying all of the Eilenberg-Steenrod axioms for cohomology.

   Suppose that E is a generalized cohomology theory. An *exterior multiplication* on E is a collection of maps

   $$\overline{\smile} : E^p(X) \otimes E^q(Y) \to E^{p+q}(X \times Y)$$

   natural in X and Y. Here naturality means that if $f : X \to X'$, $g : Y \to Y'$, then for all α, β we have

   $$f^*(\alpha) \overline{\smile} g^*(\beta) = (f \times g)^*(\alpha \overline{\smile} \beta).$$

   A *multiplication* on E is a map

   $$\smile : E^p(X) \otimes E^q(X) \to E^{p+q}(X)$$

   natural in X; i.e., if $f : X \to X'$, then

   $$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

   Show that an exterior multiplication determines an interior multiplication and vice versa. Explain what it means for a multiplication or exterior multiplication to be associative, and show that (under the correspondence between interior and exterior multiplications) that these two notions coincide. Show that if a multiplication is associative, then $E^*(X)$ is a graded ring for all X, and maps $X \to Y$ induce ring maps $E^*(Y) \to E^*(X)$.

5. An n-dimensional *manifold with boundary* is a space M such that every point has a open neighborhood homeomorphic to the open n-disc D^n or the open half-disc

$$D^n_+ = \{(x_1, \cdots, x_n) \in D^n \mid x_1 \geq 0\}.$$

The *boundary* of M, written ∂M is the subset of points that have neighborhoods homeomorphic to D^n_+, and the *interior* of M is the complement of the boundary.

Suppose M is a compact manifold with boundary. We say that it is *orientable* if there is a fundamental class $[M] \in H_n(M, \partial M)$ whose restriction to $H_n(M, M \setminus \{p\})$ is a generator for all p in the interior of M.

Show that ∂M is an $(n-1)$-dimensional manifold. If M is a compact orientable manifold with boundary, show that the boundary map $H_n(M, \partial M) \to H_{n-1}(\partial M)$ takes the fundamental class $[M]$ to a fundamental class of ∂M (i.e., show $\partial[M] = [\partial M]$).