1.51. **The distinguished invertible object.** Let $\mathcal{C}$ be a finite tensor category with classes of simple objects labeled by a set I. Since duals to projective objects are projective, we can define a map $D : I \to I$ such that $P_i^* = P_{D(i)}$. It is clear that $D^2(i) = i^{**}$.

Let 0 be the label for the unit object. Let $\rho = D(0)$. (In other words, $^*L_\rho$ is the socle of $P_0 = P(\mathbf{1})$). We have

$$\mathsf{Hom}(P_i^*, L_j) = \mathsf{Hom}(\mathbf{1}, P_i \otimes L_j) = \mathsf{Hom}(\mathbf{1}, \oplus_k N_{kj^*}^i P_k).$$

This space has dimension $N_{\rho j^*}^i$. Thus we get

$$N_{\rho j^*}^i = \delta_{D(i),j}.$$

Let now L_ρ be the corresponding simple object. By Proposition 1.47.2, we have

$$L_\rho^* \otimes P_m \cong \oplus_k N_{\rho m}^k P_k \cong P_{D(m)^*}.$$

Lemma 1.51.1. L_ρ is an invertible object.

Proof. The last equation implies that the matrix of action of L_{ρ^*} on projectives is a permutation matrix. Hence, the Frobenius-Perron dimension of L_{ρ^*} is 1, and we are done. $\square$

Lemma 1.51.2. *One has:* $P_{D(i)} = P_{*i} \otimes L_\rho$; $L_{D(i)} = L_{*i} \otimes L_\rho$.

Proof. It suffices to prove the first statement. Therefore, our job is to show that $\dim \mathsf{Hom}(P_i^*, L_j) = \dim \mathsf{Hom}(P_{*i}, L_j \otimes L_{\rho^*})$. The left hand side was computed before, it is $N_{\rho j^*}^i$. On the other hand, the right hand side is N_{j,ρ^*}^{*i} (we use that $\rho^* = {}^*\rho$ for an invertible object ρ). These numbers are equal by the properies of duality, so we are done. $\square$

Corollary 1.51.3. *One has:* $P_{i^{**}} = L_\rho^* \otimes P_{**i} \otimes L_\rho$; $L_{i^{**}} = L_\rho^* \otimes L_{**i} \otimes L_\rho$.

Proof. Again, it suffices to prove the first statement. We have

$$P_{i^{**}} = P_i^{**} = (P_{*i} \otimes L_\rho)^* = L_\rho^* \otimes P_{*i}^* = L_\rho^* \otimes P_{**i} \otimes L_\rho$$

$\square$

Definition 1.51.4. L_ρ is called the *distinguished invertible object* of $\mathcal{C}$.

We see that for any i, the socle of P_i is $\hat{L}_i := L_\rho^* \otimes^{**} L_i = L_i^{**} \otimes L_\rho^*$. This implies the following result.

Corollary 1.51.5. *Any finite dimensional quasi-Hopf algebra H is a Frobenius algebra, i.e. H is isomorphic to H^* as a left H-module.*

Proof. It is easy to see that that a Frobenius algebra is a quasi-Frobenius algebra (i.e. a finite dimensional algebra for which projective and injective modules coincide), in which the socle of every indecomposable

projective module has the same dimension as its cosocle (i.e., the simple quotient). As follows from the above, these conditions are satisfied for finite dimensional quasi-Hopf algebras (namely, the second condition follows from the fact that L_ρ is 1-dimensional). $\qquad\square$

1.52. Integrals in quasi-Hopf algebras.

Definition 1.52.1. A *left integral* in an algebra H with a counit $\varepsilon : H \to k$ is an element $I \in H$ such that $xI = \varepsilon(x)I$ for all $x \in H$. Similarly, a *right integral* in H is an element $I \in H$ such that $Ix = \varepsilon(x)I$ for all $x \in H$.

Remark 1.52.2. Let H be the convolution algebra of distributions on a compact Lie group G. This algebra has a counit ε defined by $\varepsilon(\xi) = \xi(1)$. Let dg be a left-invariant Haar measure on G. Then the distribution $I(f) = \int_G f(g)dg$ is a left integral in H (unique up to scaling). This motivates the terminology.

Note that this example makes sense for a finite group G over any field k. In this case, $H = k[G]$, and $I = \sum_{g \in G} g$ is both a left and a right integral.

Proposition 1.52.3. *Any finite dimensional quasi-Hopf algebra admits a unique nonzero left integral up to scaling and a unique nonzero right integral up to scaling.*

Proof. It suffices to prove the statement for left integrals (for right integrals the statement is obtained by applying the antipode). A left integral is the same thing as a homomorphism of left modules $k \to H$. Since H is Frobenius, this is the same as a homomorphism $k \to H^*$, i.e. a homomorphism $H \to k$. But such homomorphisms are just multiples of the counit. $\qquad\square$

Note that the space of left integrals of an algebra H with a counit is a right H-module (indeed, if I is a left integral, then so is Iy for all $y \in H$). Thus, for finite dimensional quasi-Hopf algebras, we obtain a character $\chi : H \to k$, such that $Ix = \chi(x)I$ for all $x \in H$. This character is called *the distinguished character* of H (if H is a Hopf algebra, it is commonly called the distinguished grouplike element of H^*, see [Mo]).

Proposition 1.52.4. *Let H be a finite dimensional quasi-Hopf algebra, and $\mathcal{C} = \mathsf{Rep}(H)$. Then L_ρ coincides with the distinguished character χ.*

Proof. Let I be a nonzero left integral in H. We have $xI = \varepsilon(x)I$ and $Ix = \chi(x)I$. This means that for any $V \in \mathcal{C}$, I defines a morphism from $V \otimes \chi^{-1}$ to V.

The element I belongs to the submodule P_i of H, whose socle is the trivial H-module. Thus, $P_i^* = P(\mathbf{1})$, and hence by Lemma 1.51.2, $i = \rho$. Thus, I defines a nonzero (but rank 1) morphism $P_\rho \otimes \chi^{-1} \to P_\rho$. The image of this morphism, because of rank 1, must be $L_0 = \mathbf{1}$, so $\mathbf{1}$ is a quotient of $P_\rho \otimes \chi^{-1}$, and hence χ is a quotient of P_ρ. Thus, $\chi = L_\rho$, and we are done. $\qquad\square$

Proposition 1.52.5. *The following conditions on a finite dimensional quasi-Hopf algebra H are equivalent:*
(i) H is semisimple;
(ii) $\varepsilon(I) \neq 0$ (where I is a left integral in H);
(iii) $I^2 \neq 0$;
(iv) I can be normalized to be an idempotent.

Proof. (ii) implies (i): If $\varepsilon(I) \neq 0$ then $k = \mathbf{1}$ is a direct summand in H as a left H-module. This implies that $\mathbf{1}$ is projective, hence $\mathrm{Rep}(H)$ is semisimple (Corollary 1.13.7).

(i) implies (iv): If H is semisimple, the integral is a multiple of the projector to the trivial representation, so the statement is obvious.

(iv) implies (iii): obvious.

(iii) implies (ii): clear, since $I^2 = \varepsilon(I)I$. $\qquad\square$

Definition 1.52.6. A finite tensor category $\mathcal{C}$ is *unimodular* if $L_\rho = \mathbf{1}$. A finite dimensional quasi-Hopf algebra H is unimodular if $\mathrm{Rep}(H)$ is a unimodular category, i.e. if left and right integrals in H coincide.

Remark 1.52.7. This terminology is motivated by the notion of a unimodular Lie group, which is a Lie group on which a left invariant Haar measure is also right invariant, and vice versa.

Remark 1.52.8. Obviously, every semisimple category is automatically unimodular.

Exercise 1.52.9. (i) Let H be the Nichols Hopf algebra of dimension 2^{n+1} (Example 1.24.9). Find the projective covers of simple objects, the distinguished invertible object, and show that H is not unimodular. In particular, Sweedler's finite dimensional Hopf algebra is not unimodular.

(ii) Do the same if H is the Taft Hopf algebra (Example 1.24.5).

(iii) Let $H = u_q(\mathfrak{sl}_2)$ be the small quantum group at a root of unity q of odd order (see Subsection 1.25). Show that H is unimodular, but H^* is not. Find the distinguished character of H^* (i.e., the distinguished grouplike element of H). What happens for the corresponding graded Hopf algebra $\mathrm{gr}(H)$?

1.53. **Dimensions of projective objects and degeneracy of the Cartan matrix.** The following result in the Hopf algebra case was proved by M.Lorenz [L]; our proof in the categorical setting is analogous to his.

Let $C_{ij} = [P_i : L_j]$ be the entries of the Cartan matrix of a finite tensor category $\mathcal{C}$.

Theorem 1.53.1. *Suppose that $\mathcal{C}$ is not semisimple, and admits an isomorphism of additive functors $u : \mathrm{Id} \to **$. Then the Cartan matrix C is degenerate over the ground field k.*

Proof. Let $\dim(V) = \mathrm{Tr}|_V(u)$ be the dimension function defined by the (left) categorical trace of u. This function is additive on exact sequences, so it is a linear functional on $\mathrm{Gr}(\mathcal{C})$.

On the other hand, the dimension of every projective object P with respect to this function is zero. Indeed, the dimension of P is the composition of maps $\mathbf{1} \to P \otimes P^* \to P^{**} \otimes P^* \to \mathbf{1}$, where the maps are the coevaluation, $u \otimes \mathrm{Id}$, and the evaluation. If this map is nonzero then $\mathbf{1}$ is a direct summand in $P \otimes P^*$, which is projective. Thus $\mathbf{1}$ is projective, So $\mathcal{C}$ is semisimple by Corollary 1.13.7. Contradiction.

Since the dimension of the unit object $\mathbf{1}$ is not zero, $\mathbf{1}$ is not a linear combination of projective objects in the Grothendieck group tensored with k. We are done. $\qquad\square$

2. Module categories

We have seen that the notion of a tensor category categorifies the notion of a ring. In a similar way, the notion of a module category categorifies the notion of a module over a ring. In this section we will develop a systematic theory of module categories over tensor categories. This theory is interesting by itself, but is also crucial for understanding the structure of tensor categories, similarly to how the study of modules is improtant in understanding the structure of rings.

We will begin with a discussion of module categories over general monoidal categories, and then pass to the k-linear case.

2.1. **The definition of a module category.** Let $\mathcal{C}$ be a monoidal category.

Definition 2.1.1. A *left module category* over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an *action (or tensor product) bifunctor* $\otimes^{\mathcal{M}} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a functorial *associativity isomorphism* (or *constraint*) $a^{\mathcal{M}} : (\bullet \otimes \bullet) \otimes^{\mathcal{M}} \bullet \to \bullet \otimes (\bullet \otimes^{\mathcal{M}} \bullet)$, such that the functor $\mathbf{1} \otimes^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$

is an autoequivalence, and $a^{\mathcal{M}}$ satisfies the *pentagon relation*:
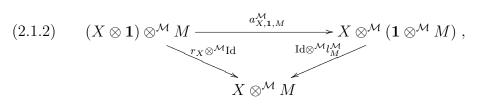(2.1.1)

$$((X \otimes Y) \otimes Z) \otimes^{\mathcal{M}} M$$

$$a^{\mathcal{M}}_{X \otimes Y, Z, M} \qquad\qquad a_{X,Y,Z} \otimes \mathrm{Id}_M$$

$$(X \otimes Y) \otimes (Z \otimes^{\mathcal{M}} M) \qquad\qquad (X \otimes (Y \otimes Z)) \otimes M$$

$$a^{\mathcal{M}}_{X,Y,Z \otimes^{\mathcal{M}} M} \Bigg\downarrow \qquad\qquad\qquad\qquad \Bigg\downarrow a^{\mathcal{M}}_{X,Y \otimes Z, M}$$

$$X \otimes (Y \otimes (Z \otimes M)) \xleftarrow{\quad \mathrm{Id}_X \otimes a^{\mathcal{M}}_{Y,Z,M} \quad} X \otimes ((Y \otimes Z) \otimes^{\mathcal{M}} M)$$

is commutative for all objects X, Y, Z in $\mathcal{C}$ and M in $\mathcal{M}$.

Clearly, this definition categorifies the notion of a module over a monoid.

In a similar way one defines a *right* $\mathcal{C}$-module category. Namely, a right $\mathcal{C}$-module category is the same thing as a left $\mathcal{C}^{op}$-module category. By a module category we will always mean a left module category unless otherwise specified.

Similarly to the case of monoidal categories, for any $\mathcal{C}$-module category $\mathcal{M}$, one has a canonical functorial *unit isomorphism* $l^{\mathcal{M}} : \mathbf{1} \otimes^{\mathcal{M}} \to \mathrm{Id}$ (also called the *unit constraint*), and one can give the following equivalent definition of a module category, making this isomorphism a part of the data.

Definition 2.1.2. A *left module category* over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with a bifunctor $\otimes^{\mathcal{M}} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, a functorial isomorphism $a^{\mathcal{M}} : (\bullet \otimes \bullet) \otimes^{\mathcal{M}} \bullet \to \bullet \otimes (\bullet \otimes^{\mathcal{M}} \bullet)$, and a functorial isomorphism $l^{\mathcal{M}} : \mathbf{1} \otimes^{\mathcal{M}} \to \mathrm{Id}$ such that $a^{\mathcal{M}}$ satisfies the pentagon relation (2.1.1), and $l^{\mathcal{M}}$ satisfies the *triangle relation*:

(2.1.2)
$$(X \otimes \mathbf{1}) \otimes^{\mathcal{M}} M \xrightarrow{\quad a^{\mathcal{M}}_{X,\mathbf{1},M} \quad} X \otimes^{\mathcal{M}} (\mathbf{1} \otimes^{\mathcal{M}} M) ,$$

$$r_X \otimes^{\mathcal{M}} \mathrm{Id} \searrow \qquad \swarrow \mathrm{Id} \otimes^{\mathcal{M}} l^{\mathcal{M}}_M$$

$$X \otimes^{\mathcal{M}} M$$

$X \in \mathcal{C}$, $M \in \mathcal{M}$.

We leave it to the reader to establish the equivalence of the two definitions; this is entirely parallel to the case of monoidal categories.

Similarly to the case of monoidal categories, one can assume without loss of generality that $\mathbf{1} \otimes^{\mathcal{M}} = \mathrm{Id}_{\mathcal{M}}$, $l^{\mathcal{M}} = \mathrm{Id}$, and we will often do so from now on. We will also often suppress the superscript $\mathcal{M}$ and write $\otimes$ instead of $\otimes^{\mathcal{M}}$.

The following proposition gives an alternative definition of a module category. Let $\mathcal{M}$ be a category. Consider the category $\mathsf{End}(\mathcal{M})$ of endofunctors of $\mathcal{M}$. As we know, $\mathsf{End}(\mathcal{M})$ is a monoidal category.

Proposition 2.1.3. *Structures of a $\mathcal{C}$-module category on $\mathcal{M}$ are in a natural 1-1 correspondence with monoidal functors $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$.*

Proof. Let $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$ be a monoidal functor. We set $X \otimes M := F(X)(M)$, and define the associativity constraint $a^{\mathcal{M}}$ using the monoidal structure on F, as a composition $(X \otimes Y) \otimes M = F(X \otimes Y)(M) \simeq F(X)(F(Y)(M)) = X \otimes (Y \otimes M)$.

Conversely, let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then for any $X \in \mathcal{C}$ we have a functor $M \mapsto X \otimes M$; thus we have a functor $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$. Using the associativity isomorphism $a^{\mathcal{M}}$, one defines a monoidal structure on F. $\square$

Exercise 2.1.4. Fill the details in the proof of Proposition 2.1.3.

Clearly, Proposition 2.1.3 categorifies the fact in elementary algebra that a module over a ring is the same thing as a representation.
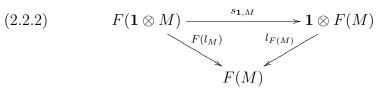
Remark 2.1.5. Note that under the correspondence of Proposition 2.1.3, the hexagon diagram for the monoidal structure on F corresponds to the pentagon diagram (2.1.1). One of the sides of the hexagon disappears due to the fact that the category $\mathsf{End}(\mathcal{M})$ is strict, so its associativity isomorphism (which is the identity) is suppressed.

Definition 2.1.6. A *module subcategory* of a $\mathcal{C}$-module category $\mathcal{M}$ is a full subcategory $\mathcal{M}' \subset \mathcal{M}$ which is closed under the action of $\mathcal{C}$.

Exercise 2.1.7. Let $\mathcal{M}$ be a $\mathcal{C}$-module category. Show that for any $X \in \mathcal{C}$ which has a left dual and any $M, N \in \mathcal{M}$ there a natural isomorphism $\mathsf{Hom}(X \otimes M, N) \simeq \mathsf{Hom}(M, {}^*X \otimes N)$. Thus, if $\mathcal{C}$ is rigid, the functor $X^* \otimes$ is left adjoint to $X \otimes$, and ${}^*X \otimes$ is right adjoint to $X \otimes$.

2.2. Module functors.

Definition 2.2.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two module categories over $\mathcal{C}$. A *module functor* from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a pair (F, s) where $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a functor, and s is a natural isomorphism $s_{X,M} : F(X \otimes M) \to X \otimes F(M)$ such that the following diagrams commute:

(2.2.1)

$$
\begin{array}{ccccc}
F(X \otimes (Y \otimes M)) & \xleftarrow{\ F(a_{X,Y,M})\ } & F((X \otimes Y) \otimes M) & \xrightarrow{\ s_{X \otimes Y, M}\ } & (X \otimes Y) \otimes F(M) \\
\downarrow{\scriptstyle s_{X,Y \otimes M}} & & & & \downarrow{\scriptstyle a_{X,Y,F(M)}} \\
X \otimes F(Y \otimes M) & & \xrightarrow{\ \mathrm{Id} \otimes s_{Y,M}\ } & & X \otimes (Y \otimes F(M))
\end{array}
$$

and

$$(2.2.2) \qquad F(\mathbf{1} \otimes M) \xrightarrow{\quad s_{\mathbf{1},M} \quad} \mathbf{1} \otimes F(M)$$

$$F(l_M) \searrow \qquad \swarrow l_{F(M)}$$

$$F(M)$$

A *module equivalence* $F : \mathcal{M}_1 \to \mathcal{M}_2$ of $\mathcal{C}$-module categories is a module functor (F, s) from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that F is an equivalence of categories.

Clearly, this definition categorifies the notion of a homomorphism (respectively, isomorphism) of modules.

Remark 2.2.2. Note that if $l^{\mathcal{M}_i} = \mathrm{Id}$ then the second diagram reduces to the condition that $s_{\mathbf{1},M} = \mathrm{Id}_{F(M)}$.

Remark 2.2.3. One can prove a version of Maclane's coherence theorem for module categories and module functors, stating that positions of brackets are, essentially, immaterial (we leave it to the reader to state and prove this theorem). For this reason, when working with module categories, we will suppress brackets from now on.

2.3. **Module categories over multitensor categories.** Our main interest will be module categories over multitensor categories (defined over a field k). In this case, we would like to consider module categories with an additional structure of an abelian category.

Let $\mathcal{C}$ be a multitensor category over k.

Definition 2.3.1. A (left or right) abelian module category over $\mathcal{C}$ is a locally finite abelian category $\mathcal{M}$ over k which is equipped with a structure of a (left or right) $\mathcal{C}$-module category, such that the functor $\otimes^{\mathcal{M}}$ is bilinear on morphisms and exact in the first variable.

Remark 2.3.2. Note that $\otimes^{\mathcal{M}}$ is always exact in the second variable due to Exercise 2.1.7.

All module categories over multitensor categories that we will consider will be abelian, so we will usually suppress the word "abelian" from now on.

Let $\mathsf{End}_L(\mathcal{M})$ be the category of left exact functors from $\mathcal{M}$ to $\mathcal{M}$. This is an abelian category. (Namely, if $\mathcal{M}$ is the category of finite dimensional comodules over a coalgebra C then $\mathsf{End}_L(\mathcal{M})$ is equivalent to a full subcategory of the category of C-bicomodules, via $F \mapsto F(C)$; note that $F(C)$ is well defined, since F, being left exact, commutes with direct limits, and thus extends to the ind-completion of $\mathcal{M}$).

Proposition 2.3.3. *Structures of a $\mathcal{C}$-module category on $\mathcal{M}$ are in a natural 1-1 correspondence with exact monoidal functors $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$.*

Proof. The proof is the same as that of Proposition 2.1.3. $\qquad\square$

We will also need to consider module functors between abelian module categories. Unless otherwise specified, we will consider only *left exact* module functors, referring to them just as "module functors".

2.4. **Direct sums.** There is a very simple construction of the direct sum of module categories.

Proposition 2.4.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two module categories over $\mathcal{C}$. Then the category $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\otimes^{\mathcal{M}} = \otimes^{\mathcal{M}_1} \oplus \otimes^{\mathcal{M}_2}$, $a^{\mathcal{M}} = a^{\mathcal{M}_1} \oplus a^{\mathcal{M}_2}$, $l^{\mathcal{M}} = l^{\mathcal{M}_1} \oplus l^{\mathcal{M}_2}$ is a module category over $\mathcal{C}$.*

Proof. Obvious. $\qquad\square$

Definition 2.4.2. The module category $\mathcal{M}$ is called the *direct sum* of module categories $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 2.4.3. We will say that a module category $\mathcal{M}$ over $\mathcal{C}$ is *indecomposable* if it is not equivalent to a nontrivial direct sum of module categories (that is, with $\mathcal{M}_1, \mathcal{M}_2$ nonzero).

2.5. **Examples of module categories.** The following are some basic examples of module categories.

Example 2.5.1. Any multitensor category $\mathcal{C}$ is a module category over itself; in this case we set $\otimes_{\mathcal{M}} = \otimes$, $a^{\mathcal{M}} = a$, $l^{\mathcal{M}} = l$. This module category can be considered as a categorification of the regular representation of an algebra.

Example 2.5.2. Let $\mathcal{C}$ be a multitensor category. Then one considers $\mathcal{M} = \mathcal{C}$ as a module category over $\mathcal{C} \boxtimes \mathcal{C}^{op}$ via $(X \boxtimes Y) \otimes_{\mathcal{M}} Z = X \otimes Z \otimes Y$. (This can be extended to the entire category $\mathcal{C} \boxtimes \mathcal{C}^{op}$ by resolving objects of this category by injective $\boxtimes$-decomposable objects).

Exercise 2.5.3. Define the associativity and unit constraints for this example using the associativity and unit constraints in $\mathcal{C}$.

This module category corresponds to the algebra considered as a bimodule over itself.

Definition 2.5.4. Let $\mathcal{C}, \mathcal{D}$ be multitensor categories. A $(\mathcal{C}, \mathcal{D})$-bimodule category is a module category over $\mathcal{C} \boxtimes \mathcal{D}^{op}$.

Example 2.5.5. Let $\mathcal{C}$ be a multitensor category and let $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}_{ij}$ be its decomposition into components (see Proposition 1.15.5). Then obviously $\mathcal{C}_{ij}$ is a $(\mathcal{C}_{ii}, \mathcal{C}_{jj})$-bimodule category.

Example 2.5.6. Let us study when the simplest category $\mathcal{M} = \text{Vec}$ is a module category over a multitensor category $\mathcal{C}$. Obviously $\text{Fun}(\mathcal{M}, \mathcal{M}) = \text{Vec}$ as a tensor category. Hence by Proposition 2.1.3 the structures of the module category over $\mathcal{C}$ on $\mathcal{M}$ are in a natural bijection with tensor functors $F : \mathcal{C} \to \text{Vec}$, that is, with fiber functors. Thus the theory of module categories can be considered as an extension of the theory of fiber functors.

Example 2.5.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. Then $\mathcal{M} = \mathcal{D}$ has a structure of module category over $\mathcal{C}$ with $X \otimes_{\mathcal{M}} Y := F(X) \otimes Y$.

Exercise 2.5.8. Define the associativity and unit constraints for this example using the tensor structure of the functor F and verify the axioms.

Example 2.5.9. Let G be a finite group and let $H \subset G$ be a subgroup. Since the restriction functor $\text{Res} : \text{Rep}(G) \to \text{Rep}(H)$ is tensor functor, we conclude that $\text{Rep}(H)$ is a module category over $\mathcal{C} = \text{Rep}(G)$. A little bit more generally, assume that we have a central extension of groups $1 \to k^{\times} \to \tilde{H} \to H \to 1$ representing an element $\psi \in H^2(H, k^{\times})$. Consider the category $\text{Rep}_{\psi}(H)$ of representations of $\tilde{H}$ such that any $\lambda \in k^{\times}$ acts by multiplication by λ (thus any object of $\text{Rep}_{\psi}(H)$ is a *projective* representation of H). Then usual tensor product and usual associativity and unit constraints determine the structure of module category over $\mathcal{C}$ on $\mathcal{M} = \text{Rep}_{\psi}(H)$. One can show that all semisimple indecomposable module categories over $\mathcal{C} = \text{Rep}(G)$ are of this form.

Example 2.5.10. Let $\mathcal{C} = \text{Vec}_G$, where G is a group. In this case, a module category $\mathcal{M}$ over $\mathcal{C}$ is an abelian category $\mathcal{M}$ with a collection of exact functors $F_g : \mathcal{M} \to \mathcal{M}$, $F_g(M) := g \otimes M$, together with a collection of functorial isomorphisms $\eta_{g,h} : F_g \circ F_h \to F_{gh}$ satisfying the 2-cocycle relation:

$$\eta_{gh,k} \circ \eta_{gh} = \eta_{g,hk} \circ \eta_{hk}$$

as morphisms $F_g \circ F_h \circ F_k \to F_{ghk}$.

Such data is called an *action* of G on $\mathcal{M}$. So, module categories over Vec_G is the same thing as abelian categories with an action of G.

Example 2.5.11. Here is an example which we consider as somewhat pathological with respect to finiteness properties: let $\mathcal{C} = \text{Vec}$ and let $\mathcal{M} = \text{Vec}$ be the category of all (possibly infinite dimensional) vector

spaces. Then the usual tensor product and the usual associativity and unit constraints determine the structure of module category over $\mathcal{C}$ on $\mathcal{M}$.

2.6. **Exact module categories for finite tensor categories.** Consider the simplest tensor category $\mathcal{C} = \text{Vec}$. Let $\mathcal{M}$ be any abelian category over k. Then $\mathcal{M}$ has a unique (up to equivalence) structure of module category over $\mathcal{C}$. Thus in this case the understanding of all module categories over $\mathcal{C}$ is equivalent to the understanding of all $k-$linear abelian categories. This seems to be too complicated even if restrict ourselves only to categories satisfying some finiteness conditions (for example, to finite categories). Thus in this section we introduce a much smaller class of module categories which is quite manageable (for example, this class admits an explicit classification for many interesting tensor categories $\mathcal{C}$) but on the other hand contains many interesting examples. Here is the main definition:

Definition 2.6.1. Let $\mathcal{C}$ be a multitensor category with enough projective objects. A module category $\mathcal{M}$ over $\mathcal{C}$ is called *exact* if for any projective object $P \in \mathcal{C}$ and any object $M \in \mathcal{M}$ the object $P \otimes M$ is projective in $\mathcal{M}$.

Exercise 2.6.2. Let $\mathcal{M}$ be an arbitrary module category over $\mathcal{C}$. Show that for any object $X \in \mathcal{C}$ and any projective object $Q \in \mathcal{M}$ the object $X \otimes Q$ is projective in $\mathcal{M}$.

It is immediate from the definition that any semisimple module category is exact (since any object in a semisimple category is projective).

Remark 2.6.3. We will see that the notion of an exact module category may be regarded as the categorical analog of the notion of a projective module in ring theory.

Example 2.6.4. Notice that in the category $\mathcal{C} = \text{Vec}$ the object $\mathbf{1}$ is projective. Therefore for an exact module category $\mathcal{M}$ over $\mathcal{C}$ any object $M = \mathbf{1} \otimes M$ is projective. Hence an abelian category $\mathcal{M}$ considered as a module category over $\mathcal{C}$ is exact if and only if it is semisimple. Thus the exact module categories over Vec are classified by the cardinality of the set of the isomorphism classes of simple objects. More generally, the same argument shows that if $\mathcal{C}$ is semisimple (and hence $\mathbf{1}$ is projective) then any exact module category over $\mathcal{C}$ is semisimple. But the classification of exact module categories over non-semisimple categories $\mathcal{C}$ can be quite nontrivial.

Example 2.6.5. Any finite multitensor category $\mathcal{C}$ considered as a module category over itself (see Example 2.5.1) is exact. Also the

category $\mathcal{C}$ considered as a module over $\mathcal{C} \boxtimes \mathcal{C}^{op}$ (see Example 2.5.2) is exact.

Example 2.6.6. Let $\mathcal{C}$ and $\mathcal{D}$ be a finite multitensor categories and let $F : \mathcal{C} \to \mathcal{D}$ be a *surjective* tensor functor. Then the category $\mathcal{D}$ considered as a module category over $\mathcal{C}$ (see Example 2.5.7) is exact by Theorem 1.49.3.

Exercise 2.6.7. Show that the assumption that F is surjective is essential for this example.

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