8. Symplectic reflection algebras

8.1. **The definition of symplectic reflection algebras.** Rational Cherednik algebras for finite Coxeter groups are a special case of a wider class of algebras called symplectic reflection algebras. To define them, let V be a finite dimensional symplectic vector space over $\mathbb{C}$ with a symplectic form ω, and G be a finite group acting symplectically (linearly) on V. For simplicity let us assume that $(\wedge^2 V^*)^G = \mathbb{C}\omega$ (i.e., V is symplectically irreducible) and that G acts faithfully on V (these assumptions are not important, and essentially not restrictive).

Definition 8.1. A symplectic reflection in G is an element g such that the rank of the operator $1 - g$ on V is 2.

If s is a symplectic reflection, then let $\omega_s(x, y)$ be the form ω applied to the projections of x, y to the image of $1 - s$ along the kernel of $1 - s$; thus ω_s is a skewsymmetric form of rank 2 on V.

Let $\mathcal{S} \subset G$ be the set of symplectic reflections, and $c : \mathcal{S} \to \mathbb{C}$ be a function which is invariant under the action of G. Let $t \in \mathbb{C}$.

Definition 8.2. The symplectic reflection algebra $\mathsf{H}_{t,c} = \mathsf{H}_{t,c}[G, V]$ is the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the ideal generated by the relation

$$[x, y] = t\omega(x, y) - 2\sum_{s\in\mathcal{S}} c_s\omega_s(x, y)s. \tag{8.1}$$

Example 8.3. Let W be a finite Coxeter group with reflection representation $\mathfrak{h}$. We can set $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\omega(x, x') = \omega(y, y') = 0$, $\omega(y, x) = (y, x)$, for $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. In this case

(1) symplectic reflections are the usual reflections in W;
(2) $\omega_s(x, x') = \omega_s(y, y') = 0$, $\omega_s(y, x) = (y, \alpha_s)(\alpha_s^\vee, x)/2$.

Thus, $\mathsf{H}_{t,c}[G, \mathfrak{h} \oplus \mathfrak{h}^*]$ coincides with the rational Cherednik algebra $H_{t,c}(G, \mathfrak{h})$ defined in Section 3.

Example 8.4. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$, and $V = \mathbb{C}^2$ be the tautological representation, with its standard symplectic form. Then all nontrivial elements of Γ are symplectic reflections, and for any symplectic reflection s, $\omega_s = \omega$. So the main commutation relation of $\mathsf{H}_{t,c}[\Gamma, V]$ takes the form

$$[y, x] = t - \sum_{g\in\Gamma, g\neq 1} 2c_g g.$$

Example 8.5. (Wreath products) Let Γ be as in the previous example, $G = \mathfrak{S}_n \ltimes \Gamma^n$, and $V = (\mathbb{C}^2)^n$. Then symplectic reflections are conjugates $(g, 1, ..., 1)$, $g \in \Gamma$, $g \neq 1$, and also conmjugates of transpositions in $\mathfrak{S}_n$ (so there is one more conjugacy class of reflections than in the previous example).

Note also that for any V, G, $\mathsf{H}_{0,0}[G, V] = G \ltimes SV$, and $\mathsf{H}_{1,0}[G, V] = G \ltimes \mathrm{W}eyl(V)$, where $\mathrm{W}eyl(V)$ is the Weyl algebra of V, i.e. the quotient of the tensor algebra $\mathbf{T}(V)$ by the relation $xy - yx = \omega(x, y)$, $x, y \in V$.

8.2. **The PBW theorem for symplectic reflection algebras.** To ensure that the symplectic reflection algebras $\mathsf{H}_{t,c}$ have good properties, we need to prove a PBW theorem for them, which is an analog of Proposition 3.5. This is done in the following theorem, which also explains the special role played by symplectic reflections.

Theorem 8.6. *Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be a linear G-equivariant function. Define the algebra H_κ to be the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the relation $[x, y] = \kappa(x, y)$, $x, y \in V$. Put an increasing filtration on H_κ by setting* $\deg(V) = 1$, $\deg(G) = 0$, *and define $\xi : \mathbb{C}G \ltimes SV \to$ $\mathrm{gr}\mathsf{H}_\kappa$ to be the natural surjective homomorphism. Then ξ is an isomorphism if and only if κ has the form*

$$\kappa(x, y) = t\omega(x, y) - 2\sum_{s\in\mathcal{S}} c_s\omega_s(x, y)s,$$

for some $t \in \mathbb{C}$ and G-invariant function $c : \mathcal{S} \to \mathbb{C}$.

Unfortunately, for a general symplectic reflection algebra we don't have a Dunkl operator representation, so the proof of the more difficult "if" part of this Theorem is not as easy as the proof of Proposition 3.5. Instead of explicit computations with Dunkl operators, it makes use of the deformation theory of Koszul algebras, which we will now discuss.

8.3. **Koszul algebras.** Let R be a finite dimensional semisimple algebra (over $\mathbb{C}$). Let A be a $\mathbb{Z}_+$-graded algebra, such that $A[0] = R$, and assume that the graded components of A are finite dimensional.

Definition 8.7. (i) The algebra A is said to be quadratic if it is generated over R by $A[1]$, and has defining relations in degree 2.

(ii) A is Koszul if all elements of $\mathrm{Ext}^i(R, R)$ (where R is the augmentation module over A) have grade degree precisely i.

Remark 8.8. (1) Thus, in a quadratic algebra, $A[2] = A[1] \otimes_R A[1]/E$, where E is the subspace (R-subbimodule) of relations.

(2) It is easy to show that a Koszul algebra is quadratic, since the condition to be quadratic is just the Koszulity condition for $i = 1, 2$.

Now let A_0 be a quadratic algebra, $A_0[0] = R$. Let E_0 be the space of relations for A_0. Let $E \subset A_0[1] \otimes_R A_0[1][[\hbar]]$ be a free (over $\mathbb{C}[[\hbar]]$) R-subbimodule which reduces to E_0 modulo $\hbar$ ("deformation of the relations"). Let A be the ($\hbar$-adically complete) algebra generated over $R[[\hbar]]$ by $A[1] = A_0[1][[\hbar]]$ with the space of defining relations E. Thus A is a $\mathbb{Z}_+$-graded algebra.

The following very important theorem is due to Beilinson, Ginzburg, and Soergel, [BGS] (less general versions appeared earlier in the works of Drinfeld [Dr], Polishchuk-Positselski [PP], Braverman-Gaitsgory [BG]). We will not give its proof.

Theorem 8.9 (Koszul deformation principle)**.** *If A_0 is Koszul then A is a topologically free $\mathbb{C}[[\hbar]]$ module if and only if so is $A[3]$.*

Remark. Note that $A[i]$ for $i < 3$ are obviously topologically free.

We will now apply this theorem to the proof of Theorem 8.6.

8.4. **Proof of Theorem 8.6.** Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be a linear G-equivariant map. We write $\kappa(x, y) = \sum_{g\in G} \kappa_g(x, y)g$, where $\kappa_g(x, y) \in \wedge^2 V^*$. To apply Theorem 8.9, let us homogenize our algebras. Namely, let $A_0 = (\mathbb{C}G \ltimes SV) \otimes \mathbb{C}[u]$ (where u has degree 1). Also let $\hbar$ be a formal parameter, and consider the deformation $A = \mathsf{H}_{\hbar u^2\kappa}$ of A_0. That is, A is the quotient of $G \ltimes \mathbf{T}(V)[u][[\hbar]]$ by the relations $[x, y] = \hbar u^2\kappa(x, y)$. This is a deformation of the type considered in Theorem 8.9, and it is easy to see that its flatness in $\hbar$ is equivalent to Theorem

8.6. Also, the algebra A_0 is Koszul, because the polynomial algebra SV is a Koszul algebra. Thus by Theorem 8.9, it suffices to show that A is flat in degree 3.

The flatness condition in degree 3 is "the Jacobi identity"

$$[\kappa(x,y),z]+[\kappa(y,z),x]+[\kappa(z,x),y]=0,$$

which must be satisfied in $\mathbb{C}G\ltimes V$. In components, this equation transforms into the system of equations

$$\kappa_g(x,y)(z-z^g)+\kappa_g(y,z)(x-x^g)+\kappa_g(z,x)(y-y^g)=0$$

for every $g\in G$ (here z^g denotes the result of the action of g on z).

This equation, in particular, implies that if x,y,g are such that $\kappa_g(x,y)\neq 0$ then for any $z\in V$ $z-z^g$ is a linear combination of $x-x^g$ and $y-y^g$. Thus $\kappa_g(x,y)$ is identically zero unless the rank of $(1-g)|_V$ is at most 2, i.e. $g=1$ or g is a symplectic reflection.

If $g=1$ then $\kappa_g(x,y)$ has to be G-invariant, so it must be of the form $t\omega(x,y)$, where $t\in\mathbb{C}$.

If g is a symplectic reflection, then $\kappa_g(x,y)$ must be zero for any x such that $x-x^g=0$. Indeed, if for such an x there had existed y with $\kappa_g(x,y)\neq 0$ then $z-z^g$ for any z would be a multiple of $y-y^g$, which is impossible since $\mathrm{Im}(1-g)|_V$ is 2-dimensional. This implies that $\kappa_g(x,y)=2c_g\omega_g(x,y)$, and c_g must be invariant.

Thus we have shown that if A is flat (in degree 3) then κ must have the form given in Theorem 8.6. Conversely, it is easy to see that if κ does have such form, then the Jacobi identity holds. So Theorem 8.6 is proved.

8.5. **The spherical subalgebra of the symplectic reflection algebra.** The properties of symplectic reflection algebras are similar to the properties of rational Cherednik algebras we have studied before. The main difference is that we no longer have the Dunkl representation and localization results, so some proofs are based on different ideas and are more complicated.

The spherical subalgebra of the symplectic reflection algebra is defined in the same way as in the Cherednik algebra case. Namely, let $\mathsf{e}=|G|^{-1}\sum_{g\in G}g$, and $\mathsf{B}_{t,c}=\mathsf{e}\mathsf{H}_{t,c}\mathsf{e}$.

Proposition 8.10. *$\mathsf{B}_{t,c}$ is commutative if and only if $t=0$.*

Proof. Let A be a $\mathbb{Z}_+$-filtered algebra. If A is not commutative, then we can define a nonzero Poisson bracket on $\mathrm{gr}A$ in the following way. Let m be the minimum of $\deg(a)+\deg(b)-\deg([a,b])$ (over $a,b\in A$ such that $[a,b]\neq 0$). Then for homogeneous elements $a_0,b_0\in A_0$ of degrees p,q, we can define $\{a_0,b_0\}$ to be the image in $A_0[p+q-m]$ of $[a,b]$, where a,b are any lifts of a_0,b_0 to A. It is easy to check that $\{\cdot,\cdot\}$ is a Poisson bracket on A_0 of degree $-m$.

Let us now apply this construction to the filtered algebra $A=\mathsf{B}_{t,c}$. We have $\mathrm{gr}(A)=A_0=(SV)^G$.

Lemma 8.11. *A_0 has a unique, up to scaling, Poisson bracket of degree -2, and no nonzero Poisson brackets of degrees <-2.*

Proof. A Poisson bracket on $(SV)^G$ is the same thing as a Poisson bracket on the variety V^*/G. On the smooth part $(V^*/G)_s$ of V^*/G, it is simply a bivector field, and we can lift it to a bivector field on the preimage V_s^* of $(V^*/G)_s$ in V^*, which is the set of points in V with trivial stabilizers. But the codimension on $V^*\setminus V_s^*$ in V^* is 2 (as $V^*\setminus V_s^*$ is a union of symplectic subspaces), so the bivector on V_s^* extends to a regular bivector on V^*. So if this bivector is homogeneous, it must have degree ≥ -2, and if it has degree -2 then it must be with constant coefficients, so being G-invariant, it is a multiple of ω. The lemma is proved. $\square$

Now, for each t, c we have a natural Poisson bracket on A_0 of degree -2, which depends linearly on t, c. So by the lemma, this bracket has to be of the form $f(t,c)\Pi$, where Π is the unique up to scaling Poisson bracket of degree -2, and f a homogeneous linear function. Thus the algebra $A = \mathsf{B}_{t,c}$ is not commutative unless $f(t,c) = 0$. On the other hand, if $f(t,c) = 0$, and $\mathsf{B}_{t,c}$ is not commutative, then, as we've shown, A_0 has a nonzero Poisson bracket of degree < -2. But By Lemma 8.11, there is no such brackets. So $\mathsf{B}_{t,c}$ must be commutative if $f(t,c) = 0$.

It remains to show that $f(t,c)$ is in fact a nonzero multiple of t. First note that $f(1,0) \neq 0$, since $\mathsf{B}_{1,0}$ is definitely noncommutative. Next, let us take a point (t,c) such that $\mathsf{B}_{t,c}$ is commutative. Look at the $\mathsf{H}_{t,c}$-module $\mathsf{H}_{t,c}\mathsf{e}$, which has a commuting action of $\mathsf{B}_{t,c}$ from the right. Its associated graded is SV as an $(\mathbb{C}G \ltimes SV, (SV)^G)$-bimodule, which implies that the generic fiber of $\mathsf{H}_{t,c}\mathsf{e}$ as a $\mathsf{B}_{t,c}$-module is the regular representation of G. So we have a family of finite dimensional representations of $\mathsf{H}_{t,c}$ on the fibers of $\mathsf{H}_{t,c}\mathsf{e}$, all realized in the regular representation of G. Computing the trace of the main commutation relation (8.1) of $\mathsf{H}_{t,c}$ in this representation, we obtain that $t = 0$ (since $\mathrm{Tr}\,(s) = 0$ for any reflection s). The proposition is proved. $\square$

Note that $\mathsf{B}_{0,c}$ has no zero divisors, since its associated graded algebra $(SV)^G$ does not. Thus, like in the Cherednik algebra case, we can define a Poisson variety M_c, the spectrum of $\mathsf{B}_{0,c}$, called the Calogero-Moser space of G, V. Moreover, the algebra $\mathsf{B}_c := \mathsf{B}_{\hbar,c}$ over $\mathbb{C}[\hbar]$ is an algebraic quantization of M_c.

8.6. **The center of the symplectic reflection algebra $\mathsf{H}_{t,c}$.** Consider the bimodule $\mathsf{H}_{t,c}\mathsf{e}$, which has a left action of $\mathsf{H}_{t,c}$ and a right commuting action of $\mathsf{B}_{t,c}$. It is obvious that $\mathrm{End}_{\mathsf{H}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} = \mathsf{B}_{t,c}$ (with opposite product). The following theorem shows that the bimodule $\mathsf{H}_{t,c}\mathsf{e}$ has the double centralizer property (i.e., $\mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} = \mathsf{H}_{t,c}$).

Note that we have a natural map $\xi_{t,c} : \mathsf{H}_{t,c} \to \mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e}$.

Theorem 8.12. *$\xi_{t,c}$ is an isomorphism for any t, c.*

Proof. The complete proof is given [EG]. We will give the main ideas of the proof skipping straightforward technical details. The first step is to show that the result is true in the graded case, $(t,c) = (0,0)$. To do so, note the following easy lemma:

Lemma 8.13. *If X is an affine complex algebraic variety with algebra of functions $\mathcal{O}_X$ and G a finite group acting freely on X then the natural map $\xi_X : G \ltimes \mathcal{O}_X \to \mathrm{End}_{\mathcal{O}_X^G}\mathcal{O}_X$ is an isomorphism.*

Therefore, the map $\xi_{0,0} : G \ltimes SV \to \mathrm{End}_{(SV)^G}(SV)$ is injective, and moreover becomes an isomorphism after localization to the field of quotients $\mathbb{C}(V^*)^G$. To show it's surjective, take $a \in \mathrm{End}_{(SV)^G}(SV)$. There exists $a' \in G \ltimes \mathbb{C}(V^*)$ which maps to a. Moreover, by Lemma 8.13, a' can have poles only at fixed points of G on V^*. But these fixed points form a subset of codimension ≥ 2, so there can be no poles and we are done in the case $(t,c) = (0,0)$.

Now note that the algebra $\mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e}$ has an increasing integer filtration (bounded below) induced by the filtration on $\mathsf{H}_{t,c}$. This is due to the fact that $\mathsf{H}_{t,c}\mathsf{e}$ is a finitely generated

$\mathsf{e}\mathsf{H}_{t,c}\mathsf{e}$-module (since it is true in the associated graded situation, by Hilbert's theorem about invariants). Also, the natural map $\mathrm{grEnd}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} \to \mathrm{End}_{\mathrm{gr}\mathsf{B}_{t,c}}\mathrm{gr}\mathsf{H}_{t,c}\mathsf{e}$ is clearly injective. Therefore, our result in the case $(t,c) = (0,0)$ implies that this map is actually an isomorphism (as so is its composition with the associated graded of $\xi_{t,c}$). Identifying the two algebras by this isomorphism, we find that $\mathrm{gr}(\xi_{t,c}) = \xi_{0,0}$. Since $\xi_{0,0}$ is an isomorphism, $\xi_{t,c}$ is an isomorphism for all t, c, as desired. [2] □

Denote by $\mathsf{Z}_{t,c}$ the center of the symplectic reflection algebra $\mathsf{H}_{t,c}$. We have the following theorem.

Theorem 8.14. *If $t \neq 0$, the center of $\mathsf{H}_{t,c}$ is trivial. If $t = 0$, we have $\mathrm{gr}\mathsf{Z}_{0,c} = \mathsf{Z}_{0,0}$. In particular, $\mathsf{H}_{0,c}$ is finitely generated over its center.*

Proof. The $t \neq 0$ case was proved by Brown and Gordon [BGo] as follows. If $t \neq 0$, we have $\mathrm{gr}\mathsf{Z}_{t,c} \subseteq \mathsf{Z}_{0,0} = (SV)^G$. Also, we have a map

$$\tau_{t,c} : \mathsf{Z}_{t,c} \longrightarrow \mathsf{B}_{t,c} = \mathsf{e}\mathsf{H}_{t,c}\mathsf{e}, \quad z \mapsto z\mathsf{e} = \mathsf{e}z\mathsf{e}.$$

The map $\tau_{t,c}$ is injective since $\mathrm{gr}(\tau_{t,c})$ is injective. In particular, the image of $\mathrm{gr}(\tau_{t,c})$ is contained in $Z(\mathsf{B}_{t,c})$, the center of $\mathsf{B}_{t,c}$. Thus it is enough to show that $Z(\mathsf{B}_{t,c})$ is trivial. To show this, note that $\mathrm{gr}Z(\mathsf{B}_{t,c})$ is contained in the Poisson center of $\mathsf{B}_{0,0}$ which is trivial. So $Z(\mathsf{B}_{t,c})$ is trivial.

Now suppose $t = 0$. We need to show that $\mathrm{gr}(\tau_{0,c}) : \mathrm{gr}(Z_{0,c}) \to Z_{0,0}$ is an isomorphism. It suffices to show that $\tau_{0,c}$ is an isomorphism. To show this, we construct $\tau_{0,c}^{-1} : \mathsf{B}_{0,c} \to \mathsf{Z}_{0,c}$ as follows.

For any $b \in \mathsf{B}_{0,c}$, since $\mathsf{B}_{0,c}$ is commutative, we have an element $\tilde{b} \in \mathrm{End}_{\mathsf{B}_{0,c}}(\mathsf{H}_{0,c}\mathsf{e})$ which is defined as the right multiplication by b. From Theorem 8.12, $\tilde{b} \in \mathsf{H}_{0,c}$. Moreover, $\tilde{b} \in \mathsf{Z}_{0,c}$ since it commutes with $\mathsf{H}_{0,c}$ as a linear operator on the faithful $\mathsf{H}_{0,c}$-module $\mathsf{H}_{0,c}\mathsf{e}$. So $\tilde{b} \in \mathsf{Z}_{0,c}$. It is easy to see that $\tilde{b}\mathsf{e} = b$. So we can set $\tilde{b} = \tau_{0,c}^{-1}(b)$ which defines the inverse map to $\tau_{0,c}$. □

8.7. **A review of deformation theory.** Now we would like to explain that symplectic reflection algebras are the most general deformations of algebras of the from $G \ltimes \mathrm{W}eyl(V)$. Before we do so, we give a brief review of classical deformation theory of associative algebras.

8.7.1. *Formal deformations of associative algebras.* Let A_0 be an associative algebra with unit over $\mathbb{C}$. Denote by μ_0 the multiplication in A_0.

Definition 8.15. A (flat) formal n-parameter deformation of A_0 is an algebra A over $\mathbb{C}[[\hbar]] = \mathbb{C}[[\hbar_1, \ldots, \hbar_n]]$ which is topologically free as a $\mathbb{C}[[\hbar]]$-module, together with an algebra isomorphism $\eta_0 : A/\mathfrak{m}A \to A_0$ where $\mathfrak{m} = \langle \hbar_1, \ldots, \hbar_n \rangle$ is the maximal ideal in $\mathbb{C}[[\hbar]]$.

When no confusion is possible, we will call A a deformation of A_0 (omitting "formal").

Let us restrict ourselves to one-parameter deformations with parameter $\hbar$. Let us choose an identification $\eta : A \to A_0[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules, such that $\eta = \eta_0$ modulo $\hbar$. Then the

[2] Here we use the fact that the filtration is bounded from below. In the case of an unbounded filtration, it is possible for a map not to be an isomorphism if its associated graded is an isomorphism. An example of this is the operator of multiplication by $1 + t^{-1}$ in the space of Laurent polynomials in t, filtered by degree.

product in A is completely determined by the product of elements of A_0, which has the form of a "star-product"

$$\mu(a,b) = a * b = \mu_0(a,b) + \hbar\mu_1(a,b) + \hbar^2\mu_2(a,b) + \cdots,$$

where $\mu_i : A_0 \otimes A_0 \to A_0$ are linear maps, and $\mu_0(a,b) = ab$.

8.7.2. *Hochschild cohomology.* The main tool in deformation theory of associative algebras is Hochschild cohomology. Let us recall its definition.

Let A be an associative algebra. Let M be a bimodule over A. A Hochschild n-cochain of A with coefficients in M is a linear map $A^{\otimes n} \to M$. The space of such cochains is denoted by $C^n(A,M)$. The differential $\mathrm{d} : C^n(A,M) \to C^{n+1}(A,M)$ is defined by the formula

$$\begin{aligned} \mathrm{d}f(a_1,\ldots,a_{n+1}) &= f(a_1,\ldots,a_n)a_{n+1} - f(a_1,\ldots,a_na_{n+1}) + f(a_1,\ldots,a_{n-1}a_n,a_{n+1}) \\ &\quad - \cdots + (-1)^n f(a_1a_2,\ldots,a_{n+1}) + (-1)^{n+1}a_1f(a_2,\ldots,a_{n+1}). \end{aligned}$$

It is easy to show that $\mathrm{d}^2 = 0$.

Definition 8.16. The Hochschild cohomology $\mathrm{HH}^\bullet(A,M)$ is defined to be the cohomology of the complex $(C^\bullet(A,M),\mathrm{d})$.

Proposition 8.17. *One has a natural isomorphism*

$$\mathrm{HH}^i(A,M) \to \mathrm{Ext}^i_{A-\mathrm{bimod}}(A,M),$$

where $A-$bimod *denotes the category of A-bimodules.*

Proof. The proof is obtained immediately by considering the bar resolution of the bimodule A:

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A,$$

where the bimodule structure on $A^{\otimes n}$ is given by

$$b(a_1 \otimes a_2 \otimes \cdots \otimes a_n)c = ba_1 \otimes a_2 \otimes \cdots \otimes a_nc,$$

and the map $\partial_n : A^{\otimes n} \to A^{\otimes n-1}$ is given by the formula

$$\partial_n(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = a_1a_2 \otimes \cdots \otimes a_n - \cdots + (-1)^n a_1 \otimes \cdots \otimes a_{n-1}a_n.$$

□

Note that we have the associative Yoneda product

$$\mathrm{HH}^i(A,M) \otimes \mathrm{HH}^j(A,N) \to \mathrm{HH}^{i+j}(A,M \otimes_A N),$$

induced by tensoring of cochains.

If $M = A$, the algebra itself, then we will denote $\mathrm{HH}^\bullet(A,M)$ by $\mathrm{HH}^\bullet(A)$. For example, $\mathrm{HH}^0(A)$ is the center of A, and $\mathrm{HH}^1(A)$ is the quotient of the Lie algebra of derivations of A by inner derivations. The Yoneda product induces a graded algebra structure on $\mathrm{HH}^\bullet(A)$; it can be shown that this algebra is supercommutative.

8.7.3. *Hochschild cohomology and deformations.* Let A_0 be an algebra, and let us look for 1-parameter deformations $A = A_0[[\hbar]]$ of A_0. Thus, we look for such series μ which satisfy the associativity equation, modulo the automorphisms of the $\mathbb{C}[[\hbar]]$-module $A_0[[\hbar]]$ which are the identity modulo $\hbar$. [3]

The associativity equation $\mu \circ (\mu \otimes \mathrm{Id}) = \mu \circ (\mathrm{Id} \otimes \mu)$ reduces to a hierarchy of linear equations:

$$\sum_{s=0}^{N} \mu_s(\mu_{N-s}(a,b),c) = \sum_{s=0}^{N} \mu_s(a, \mu_{N-s}(b,c)).$$

(These equations are linear in μ_N if μ_i, $i < N$, are known).

To study these equations, one can use Hochschild cohomology. Namely, we have the following standard facts (due to Gerstenhaber, [Ge]), which can be checked directly.

(1) The linear equation for μ_1 says that μ_1 is a Hochschild 2-cocycle. Thus algebra structures on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the space $Z^2(A_0)$ of Hochschild 2-cocycles of A_0 with values in $M = A_0$.

(2) If μ_1, μ_1' are two 2-cocycles such that $\mu_1 - \mu_1'$ is a coboundary, then the algebra structures on $A_0[\hbar]/\hbar^2$ corresponding to μ_1 and μ_1' are equivalent by a transformation of $A_0[\hbar]/\hbar^2$ that equals the identity modulo $\hbar$, and vice versa. Thus equivalence classes of multiplications on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the cohomology $\mathrm{HH}^2(A_0)$.

(3) The linear equation for μ_N says that $d\mu_N$ is a certain quadratic expression b_N in $\mu_1, \ldots, \mu_{N-1}$. This expression is always a Hochschild 3-cocycle, and the equation is solvable if and only if it is a coboundary. Thus the cohomology class of b_N in $\mathrm{HH}^3(A_0)$ is the only obstruction to solving this equation.

8.7.4. *Universal deformation.* In particular, if $\mathrm{HH}^3(A_0) = 0$ then the equation for μ_n can be solved for all n, and for each n the freedom in choosing the solution, modulo equivalences, is the space $H = \mathrm{HH}^2(A_0)$. Thus there exists an algebra structure over $\mathbb{C}[[H]]$ on the space $A_u := A_0[[H]]$ of formal functions from H to A_0, $a, b \mapsto \mu_u(a,b) \in A_0[[H]]$, $(a, b \in A_0)$, such that $\mu_u(a,b)(0) = ab \in A_0$, and every 1-parameter flat formal deformation A of A_0 is given by the formula $\mu(a,b)(\hbar) = \mu_u(a,b)(\gamma(\hbar))$ for a unique formal series $\gamma \in \hbar H[[\hbar]]$, with the property that $\gamma'(0)$ is the cohomology class of the cocycle μ_1.

Such an algebra A_u is called a *universal deformation* of A_0. It is unique up to an isomorphism (which may involve an automorphism of $\mathbb{C}[[H]]$). [4]

Thus in the case $\mathrm{HH}^3(A_0) = 0$, deformation theory allows us to completely classify 1-parameter flat formal deformations of A_0. In particular, we see that the "moduli space" parametrizing formal deformations of A_0 is a smooth space – it is the formal neighborhood of zero in H.

If $\mathrm{HH}^3(A_0)$ is nonzero then in general the universal deformation parametrized by H does not exist, as there are obstructions to deformations. In this case, the moduli space of

[3]Note that we don't have to worry about the existence of a unit in A since a formal deformation of an algebra with unit always has a unit.

[4]In spite of the universal property of A_u, it may happen that there is an isomorphism between the algebras A^1 and A^2 corresponding to different paths $\gamma_1(\hbar), \gamma_2(\hbar)$ (of course, reducing to a nontrivial automorphism of A_0 modulo $\hbar$). For this reason, sometimes A_u is called a *semiuniversal*, rather than universal, deformation of A_0.

deformations will be a closed subscheme of the formal neighborhood of zero in H, which is often singular. On the other hand, even when $\mathrm{HH}^3(A_0) \neq 0$, the universal deformation parametrized by (the formal neighborhood of zero in) H may exist (although its existence may be more difficult to prove than in the vanishing case). In this case one says that the deformations of A_0 are unobstructed (since all obstructions vanish even though the space of obstructions doesn't).

8.8. **Deformation-theoretic interpretation of symplectic reflection algebras.** Let V be a symplectic vector space (over $\mathbb{C}$) and $\mathrm{W}eyl(V)$ the Weyl algebra of V. Let G be a finite group acting symplectically on V. Then from the definition, we have

$$A_0 := \mathsf{H}_{1,0}[G,V] = G \ltimes \mathrm{W}eyl(V).$$

Let us calculate the Hochschild cohomology of this algebra.

Theorem 8.18 (Alev, Farinati, Lambre, Solotar, [AFLS])**.** *The cohomology space $\mathrm{HH}^i(G \ltimes \mathrm{W}eyl(V))$ is naturally isomorphic to the space of conjugation invariant functions on the set S_i of elements $g \in G$ such that* $\operatorname{rank}(1-g)|_V = i$*.*

Corollary 8.19. *The odd cohomology of $G \ltimes \mathrm{W}eyl(V)$ vanishes, and $\mathrm{HH}^2(G \ltimes \mathrm{W}eyl(V))$ is the space $\mathbb{C}[\mathcal{S}]^G$ of conjugation invariant functions on the set of symplectic reflections. In particular, there exists a universal deformation A of $A_0 = G \ltimes \mathrm{W}eyl(V)$ parametrized by $\mathbb{C}[\mathcal{S}]^G$.*

Proof. Directly from the theorem. □

Proof of Theorem 8.18.

Lemma 8.20. *Let B be a $\mathbb{C}$-algebra together with an action of a finite group G. Then*

$$\mathrm{HH}^*(G \ltimes B, G \ltimes B) = (\bigoplus_{g\in G} \mathrm{HH}^*(B, Bg))^G,$$

where Bg is the bimodule isomorphic to B as a space where the left action of B is the usual one and the right action is the usual action twisted by g.

Proof. The algebra $G \ltimes B$ is a projective B-module. Therefore, using the Shapiro lemma, we get

$$\begin{aligned} & \mathrm{HH}^*(G \ltimes B, G \ltimes B) = \mathrm{Ext}^*_{(G\times G)\ltimes(B\otimes B^{op})}(G \ltimes B, G \ltimes B) \\ = \; & \mathrm{Ext}^*_{G_{diagonal}\ltimes(B\otimes B^{op})}(B, G \ltimes B) = \mathrm{Ext}^*_{B\otimes B^{op}}(B, G \ltimes B)^G \\ = \; & (\bigoplus_{g\in G} \mathrm{Ext}^*_{B\otimes B^{op}}(B, Bg))^G = (\bigoplus_{g\in G} \mathrm{HH}^*(B, Bg))^G, \end{aligned}$$

as desired. □

Now apply the lemma to $B = \mathrm{W}eyl(V)$. For this we need to calculate $\mathrm{HH}^*(B, Bg)$, where g is any element of G. We may assume that g is diagonal in some symplectic basis: $g = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1})$. Then by the Künneth formula we find that

$$\mathrm{HH}^*(B, Bg) = \bigotimes_{i=1}^n \mathrm{HH}^*(\mathbb{A}_1, \mathbb{A}_1 g_i),$$

where $\mathbb{A}_1$ is the Weyl algebra of the 2-dimensional space, (generated by x, y with defining relation $xy - yx = 1$), and $g_i = \mathrm{diag}(\lambda_i, \lambda_i^{-1})$.

Thus we need to calculate $\mathrm{HH}^*(B, Bg)$, $B = \mathbb{A}_1$, $g = \mathrm{diag}(\lambda, \lambda^{-1})$.

Proposition 8.21. *$\mathrm{HH}^*(B, Bg)$ is 1-dimensional, concentrated in degree 0 if $\lambda = 1$ and in degree 2 otherwise.*

Proof. If $B = \mathbb{A}_1$ then B has the following Koszul resolution as a B-bimodule:

$$B \otimes B \to B \otimes \mathbb{C}^2 \otimes B \to B \otimes B \to B.$$

Here the first map is given by the formula

$$b_1 \otimes b_2 \quad \mapsto \quad b_1 \otimes x \otimes yb_2 - b_1 \otimes y \otimes xb_2 - b_1 y \otimes x \otimes b_2 + b_1 x \otimes y \otimes b_2,$$

the second map is given by

$$b_1 \otimes x \otimes b_2 \mapsto b_1 x \otimes b_2 - b_1 \otimes xb_2, \quad b_1 \otimes y \otimes b_2 \mapsto b_1 y \otimes b_2 - b_1 \otimes yb_2,$$

and the third map is the multiplication.

Thus the cohomology of B with coefficients in Bg can be computed by mapping this resolution into Bg and taking the cohomology. This yields the following complex $C^\bullet$:

$$0 \to Bg \to Bg \oplus Bg \to Bg \to 0, \tag{8.2}$$

where the first nontrivial map is given by $bg \mapsto [bg, y] \otimes x - [bg, x] \otimes y$, and the second nontrivial map is given by $bg \otimes x \mapsto [x, bg]$, $bg \otimes y \mapsto [y, bg]$.

Consider first the case $g = 1$. Equip the complex $C^\bullet$ with the Bernstein filtration ($\deg(x) = \deg(y) = 1$), starting with $0, 1, 2$, for C^0, C^1, C^2, respectively (this makes the differential preserve the filtration). Consider the associated graded complex $C^\bullet_{\mathrm{gr}}$. In this complex, brackets are replaced with Poisson brackets, and thus it is easy to see that $C^\bullet_{\mathrm{gr}}$ is the De Rham complex for the affine plane, so its cohomology is $\mathbb{C}$ in degree 0 and 0 in other degrees. Therefore, the cohomology of $C^\bullet$ is the same.

Now consider $g \neq 1$. In this case, declare that C^0, C^1, C^2 start in degrees 2,1,0 respectively (which makes the differential preserve the filtration), and again consider the graded complex $C^\bullet_{\mathrm{gr}}$. The graded Euler characteristic of this complex is $(t^2 - 2t + 1)(1 - t)^{-2} = 1$.

The cohomology in the C^0_{gr} term is the set of $b \in \mathbb{C}[x, y]$ such that $ab = ba^g$ for all a. This means that $\mathrm{HH}^0 = 0$.

The cohomology of the C^2_{gr} term is the quotient of $\mathbb{C}[x, y]$ by the ideal generated by $a - a^g$, $a \in \mathbb{C}[x, y]$. Thus the cohomology HH^2 of the rightmost term is 1-dimensional, in degree 0. By the Euler characteristic argument, this implies that $\mathrm{HH}^1 = 0$. The cohomology of the filtered complex $C^\bullet$ is therefore the same, and we are done. $\square$

The proposition implies that in the n-dimensional case $\mathrm{HH}^*(B, Bg)$ is 1-dimensional, concentrated in degree $\mathrm{rank}\,(1 - g)$. It is not hard to check that the group G acts on the sum of these 1-dimensional spaces by simply permuting the basis vectors. Thus the theorem is proved. $\square$

Remark 8.22. Another proof of Theorem 8.18 is given in [Pi].

Theorem 8.23. *The algebra $\mathsf{H}_{1,c}[G, V]$, with formal c, is the universal deformation of $\mathsf{H}_{1,0}[G, V] = G \ltimes \mathrm{Weyl}(V)$. More specifically, the map $f : \mathbb{C}[\mathcal{S}]^G \to \mathrm{HH}^2(G \ltimes \mathrm{Weyl}(V))$ induced by this deformation coincides with the isomorphism of Corollary 8.19.*

Proof. The proof (which we will not give) can be obtained by a direct computation with the Koszul resolution for $G \ltimes \mathrm{W}eyl(V)$. Such a proof is given in [Pi]. The paper [EG] proves a slightly weaker statement that the map f is an isomorphism, which suffices to show that $\mathsf{H}_{1,c}(G,V)$ is the universal deformation of $\mathsf{H}_{1,0}[G,V]$. □

8.9. **Finite dimensional representations of $\mathsf{H}_{0,c}$.** Let $\mathsf{M}_c = \mathrm{Spec}\mathsf{Z}_{0,c}$. We can regard $\mathsf{H}_{0,c} = \mathsf{H}_{0,c}[G,V]$ as a finitely generated module over $\mathsf{Z}_{0,c} = \mathcal{O}(\mathsf{M}_c)$. Let $\chi \in \mathsf{M}_c$ be a central character, $\chi : \mathsf{Z}_{0,c} \to \mathbb{C}$. Denote by $\langle\chi\rangle$ the ideal in $\mathsf{H}_{0,c}$ generated by the kernel of χ.

Proposition 8.24. *If χ is generic then $\mathsf{H}_{0,c}/\langle\chi\rangle$ is the matrix algebra of size $|G|$. In particular, $\mathsf{H}_{0,c}$ has a unique irreducible representation V_χ with central character χ. This representation is isomorphic to $\mathbb{C}G$ as a G-module.*

Proof. It is shown by a standard argument (which we will skip) that it is sufficient to check the statement in the associated graded case $c = 0$. In this case, for generic χ, $G \ltimes SV/\langle\chi\rangle = G \ltimes \mathrm{F}un(\mathcal{O}_\chi)$, where $\mathcal{O}_\chi$ is the (free) orbit of G consisting of the points of V^* that map to $\chi \in V^*/G$, and $\mathrm{F}un(\mathcal{O}_\chi)$ is the algebra of functions on $\mathcal{O}_\chi$. It is easy to see that this algebra is isomorphic to a matrix algebra, and has a unique irreducible representation, $\mathrm{F}un(\mathcal{O}_\chi)$, which is a regular representation of G. □

Corollary 8.25. *Any irreducible representation of $\mathsf{H}_{0,c}$ has dimension $\le |G|$.*

Proof. We will use the following lemma.

Lemma 8.26 (The Amitsur-Levitzki identity)**.** *For any $N \times N$ matrices $X_1, \dots, X_{2N}$ with entries in a commutative ring A,*

$$\sum_{\sigma \in \mathfrak{S}_{2n}} (-1)^\sigma X_{\sigma(1)} \cdots X_{\sigma(2N)} = 0.$$

Proof. Consider the ring $\mathrm{Mat}_N(A) \otimes \wedge(\xi_1, \dots, \xi_{2n})$. Let $X = \sum_i X_i \xi_i \in R$. So we have

$$X^2 = \sum_{i<j} [X_i, X_j]\xi_i\xi_j \in \mathrm{Mat}_N(A \otimes \wedge^{\mathrm{even}}(\xi_1, \dots, \xi_{2n})).$$

It is obvious that $\mathrm{Tr}\, X^2 = 0$. Similarly, one can easily show that $\mathrm{Tr}\, X^4 = 0, \dots, \mathrm{Tr}\, X^{2N} = 0$. Since the ring $A \otimes \wedge^{\mathrm{even}}(\xi_1, \dots, \xi_{2n})$ is commutative, from the Cayley-Hamilton theorem, we know that $X^{2N} = 0$ which implies the lemma. □

Since for generic χ the algebra $\mathsf{H}_{0,c}/\langle\chi\rangle$ is a matrix algebra, the algebra $\mathsf{H}_{0,c}$ satisfies the Amitsur-Levitzki identity. Next, note that since $\mathsf{H}_{0,c}$ is a finitely generated $\mathsf{Z}_{0,c}$-module (by passing to the associated graded and using Hilbert's theorem), every irreducible representation of $\mathsf{H}_{0,c}$ is finite dimensional. If $\mathsf{H}_{0,c}$ had an irreducible representation E of dimension $m > |G|$, then by the density theorem the matrix algebra Mat_m would be a quotient of $\mathsf{H}_{0,c}$. But one can show that the Amitsur-Levitzki identity of degree $|G|$ is not satisfied for matrices of bigger size than $|G|$. Contradiction. Thus, $\dim E \le |G|$, as desired. □

In general, for special central characters there are representations of $\mathsf{H}_{0,c}$ of dimension less than $|G|$. However, in some cases one can show that all irreducible representations have dimension exactly $|G|$. For example, we have the following result.

Theorem 8.27. *Let $G = \mathfrak{S}_n$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ (the rational Cherednik algebra for $\mathfrak{S}_n$). Then for $c \neq 0$, every irreducible representation of $\mathsf{H}_{0,c}$ has dimension $n!$ and is isomorphic to the regular representation of $\mathfrak{S}_n$.*

Proof. Let E be an irreducible representation of $\mathsf{H}_{0,c}$. Let us calculate the trace in E of any permutation $\sigma \neq 1$. Let j be an index such that $\sigma(j) = i \neq j$. Then $s_{ij}\sigma(j) = j$. Hence in $\mathsf{H}_{0,c}$ we have

$$[y_j, x_i s_{ij}\sigma] = [y_j, x_i]s_{ij}\sigma = cs_{ij}^2\sigma = c\sigma.$$

Hence $\operatorname{Tr}_E(\sigma) = 0$, and thus E is a multiple of the regular representation of $\mathfrak{S}_n$. But by Theorem 8.25, $\dim E \leq n!$, so we get that E is the regular representation, as desired. □

8.10. **Azumaya algebras.** Let Z be a finitely generated commutative algebra over $\mathbb{C}$, $M = \operatorname{Spec} Z$ the corresponding affine scheme, and A a finitely generated Z-algebra.

Definition 8.28. A is said to be an Azumaya algebra of degree N if the completion $\hat{A}_\chi$ of A at every maximal ideal χ in Z is the matrix algebra of size N over the completion $\hat{Z}_\chi$ of Z.

Thus, an Azumaya algebra should be thought of as a bundle of matrix algebras on M. [5] For example, if E is an algebraic vector bundle on M then $\operatorname{End}(E)$ is an Azumaya algebra. However, not all Azumaya algebras are of this form.

Example 8.29. For $q \in \mathbb{C}^*$, consider the quantum torus

$$\mathbb{T}_q = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}\rangle / \langle XY - qYX\rangle.$$

If q is a root of unity of order N, then the center of $\mathbb{T}_q$ is $\langle X^{\pm N}, Y^{\pm N}\rangle = \mathbb{C}[M]$ where $M = (\mathbb{C}^*)^2$. It is not difficult to show that $\mathbb{T}_q$ is an Azumaya algebra of degree N, but $\mathbb{T}_q \otimes_{\mathbb{C}[M]} \mathbb{C}(M) \not\cong \operatorname{Mat}_N(\mathbb{C}(M))$, so $\mathbb{T}_q$ is not the endomorphism algebra of a vector bundle.

Example 8.30. Let X be a smooth irreducible variety over a field of characteristic p. Then $\mathcal{D}(X)$, the algebra of differential operators on X, is an Azumaya algebra with rank $p^{\dim X}$, which is not an endomorphism algebra of a vector bundle. Its center is $Z = \mathcal{O}(T^*X)^{\mathbb{F}}$, the Frobenius twisted functions on T^*X.

It is clear that if A is an Azumaya algebra (say, over $\mathbb{C}$) then for every central character χ of A, $A/\langle\chi\rangle$ is the algebra $\operatorname{Mat}_N(\mathbb{C})$ of complex N by N matrices, and every irreducible representation of A has dimension N.

The following important result is due to M. Artin.

Theorem 8.31. *Let A be a finitely generated (over $\mathbb{C}$) polynomial identity (PI) algebra of degree N (i.e. all the polynomial relations of the matrix algebra of size N are satisfied in A). Then A is an Azumaya algebra if and only if every irreducible representation of A has dimension exactly N.*

Proof. See [Ar] Theorem 8.3. □

[5] If M is not affine, one can define, in a standard manner, the notion of a sheaf of Azumaya algebras on M.

Thus, by Theorem 8.27, for $G = \mathfrak{S}_n$, the rational Cherednik algebra $H_{0,c}(\mathfrak{S}_n, \mathbb{C}^n)$ for $c \neq 0$ is an Azumaya algebra of degree $n!$. Indeed, this algebra is PI of degree $n!$ because the classical Dunkl representation embeds it into matrices of size $n!$ over $\mathbb{C}(x_1, \ldots, x_n, p_1, \ldots, p_n)^{\mathfrak{S}_n}$.

Let us say that $\chi \in M$ is an Azumaya point if for some affine neighborhood U of χ the localization of A to U is an Azumaya algebra. Obviously, the set $\mathrm{Az}(M)$ of Azumaya points of M is open.

Now we come back to the study the space M_c corresponding to a symplectic reflection algebra $\mathsf{H}_{0,c}$.

Theorem 8.32. *The set* $\mathrm{Az}(\mathsf{M}_c)$ *coincides with the set of smooth points of* M_c.

The proof of this theorem is given in the following two subsections.

Corollary 8.33. *If* $G = \mathfrak{S}_n$ *and* $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ *(the rational Cherednik algebra case) then* M_c *is a smooth algebraic variety for* $c \neq 0$.

Proof. Directly from the above theorem. □

8.11. **Cohen-Macaulay property and homological dimension.** To prove Theorem 8.32, we will need some commutative algebra tools. Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors. By Noether's normalization lemma, there exist elements $z_1, \ldots, z_n \in Z$ which are algebraically independent, such that Z is a finitely generated module over $\mathbb{C}[z_1, \ldots, z_n]$.

Definition 8.34. The algebra Z (or the variety $\mathrm{Spec} Z$) is said to be Cohen-Macaulay if Z is a locally free (=projective) module over $\mathbb{C}[z_1, \ldots, z_n]$. [6]

Remark 8.35. It was shown by Serre that if Z is locally free over $\mathbb{C}[z_1, \ldots, z_n]$ for some choice of $z_1, \ldots, z_n$, then it happens for any choice of them (such that Z is finitely generated as a module over $\mathbb{C}[z_1, \ldots, z_n]$).

Remark 8.36. Another definition of the Cohen-Macaulay property is that the dualizing complex $\omega_Z^\bullet$ of Z is concentrated in degree zero. We will not discuss this definition here.

It can be shown that the Cohen-Macaulay property is stable under localization. Therefore, it makes sense to make the following definition.

Definition 8.37. An algebraic variety X is Cohen-Macaulay if the algebra of functions on every affine open set in X is Cohen-Macaulay.

Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors, and let M be a finitely generated module over Z.

Definition 8.38. M is said to be Cohen-Macaulay if for some algebraically independent $z_1, \ldots, z_n \in Z$ such that Z is finitely generated over $\mathbb{C}[z_1, \ldots, z_n]$, M is locally free over $\mathbb{C}[z_1, \ldots, z_n]$.

Again, if this happens for some $z_1, \ldots, z_n$, then it happens for any of them. We also note that M can be Cohen-Macaulay without Z being Cohen-Macaulay, and that Z is a Cohen-Macaulay algebra iff it is a Cohen-Macaulay module over itself.

We will need the following standard properties of Cohen-Macaulay algebras and modules.

[6] It was proved by Quillen that a locally free module over a polynomial algebra is free; this is a difficult theorem, which will not be needed here.

Theorem 8.39. (i) *Let $Z_1 \subset Z_2$ be a finite extension of finitely generated commutative $\mathbb{C}$-algebras, without zero divisors, and M be a finitely generated module over Z_2. Then M is Cohen-Macaulay over Z_2 iff it is Cohen-Macaulay over Z_1.*

(ii) *Suppose that Z is the algebra of functions on a smooth affine variety. Then a Z-module M is Cohen-Macaulay if and only if it is projective.*

Proof. The proof can be found in the text book [Ei]. □

In particular, this shows that the algebra of functions on a smooth affine variety is Cohen-Macaulay. Algebras of functions on many singular varieties are also Cohen-Macaulay.

Example 8.40. The algebra of regular functions on the cone $xy = z^2$ is Cohen-Macaulay. This algebra can be identified as $\mathbb{C}[a, b]^{\mathbb{Z}_2}$ by letting $x = a^2, y = b^2$ and $z = ab$, where the $\mathbb{Z}_2$ action is defined by $a \mapsto -a$, $b \mapsto -b$. It contains a subalgebra $\mathbb{C}[a^2, b^2]$, and as a module over this subalgebra, it is free of rank 2 with generators 1, ab.

Example 8.41. Any irreducible affine algebraic curve is Cohen-Macaulay. For example, the algebra of regular functions on $y^2 = x^3$ is isomorphic to the subalgebra of $\mathbb{C}[t]$ spanned by $1, t^2, t^3, \ldots$. It contains a subalgebra $\mathbb{C}[t^2]$ and as a module over this subalgebra, it is free of rank 2 with generators 1, t^3.

Example 8.42. Consider the subalgebra in $\mathbb{C}[x, y]$ spanned by 1 and $x^i y^j$ with $i+j \geq 2$. It is a finite generated module over $\mathbb{C}[x^2, y^2]$, but not free. So this algebra is not Cohen-Macaulay.

Another tool we will need is homological dimension. We will say that an algebra A has homological dimension $\leq d$ if any (left) A-module M has a projective resolution of length $\leq d$. The homological dimension of A is the smallest integer having this property. If such an integer does not exist, A is said to have infinite homological dimension.

It is easy to show that the homological dimension of A is $\leq d$ if and only if for any A-modules M, N one has $\mathrm{Ext}^i(M, N) = 0$ for $i > d$. Also, the homological dimension clearly does not decrease under taking associated graded of the algebra under a positive filtration (this is clear from considering the spectral sequence attached to the filtration).

It follows immediately from this definition that homological dimension is Morita invariant. Namely, recall that a Morita equivalence between algebras A and B is an equivalence of categories A-mod $\to$ B-mod. Such an equivalence maps projective modules to projective ones, since projectivity is a categorical property (P is projective if and only if the functor $\mathrm{Hom}(P, \cdot)$ is exact). This implies that if A and B are Morita equivalent then their homological dimensions are the same.

Then we have the following important theorem.

Theorem 8.43. *The homological dimension of a commutative finitely generated $\mathbb{C}$-algebra Z is finite if and only if Z is regular, i.e. is the algebra of functions on a smooth affine variety.*

8.12. **Proof of Theorem 8.32.** First let us show that any smooth point χ of M_c is an Azumaya point. Since $\mathsf{H}_{0,c} = \mathrm{End}_{\mathsf{B}_{0,c}} \mathsf{H}_{0,c}\mathsf{e} = \mathrm{End}_{\mathsf{Z}_{0,c}}(\mathsf{H}_{0,c}\mathsf{e})$, it is sufficient to show that the coherent sheaf on M_c corresponding to the module $\mathsf{H}_{0,c}\mathsf{e}$ is a vector bundle near χ. By Theorem 8.39 (ii), for this it suffices to show that $\mathsf{H}_{0,c}\mathsf{e}$ is a Cohen-Macaulay $\mathsf{Z}_{0,c}$-module.

To do so, first note that the statement is true for $c = 0$. Indeed, in this case the claim is that SV is a Cohen-Macaulay module over $(SV)^G$. But SV is a polynomial algebra, which is Cohen-Macaulay, so the result follows from Theorem 8.39, (i).

Now, we claim that if Z, M are positively filtered and $\mathrm{gr}M$ is a Cohen-Macaulay $\mathrm{gr}Z$-module then M is a Cohen-Macaulay Z-module. Indeed, let $z_1, \ldots, z_n$ be homogeneous algebraically independent elements of $\mathrm{gr}Z$ such that $\mathrm{gr}Z$ is a finite module over the subalgebra generated by them. Let $z_1', \ldots, z_n'$ be their liftings to Z. Then $z_1', \ldots, z_n'$ are algebraically independent, and the module M over $\mathbb{C}[z_1', \ldots, z_n']$ is finitely generated and (locally) free since so is the module $\mathrm{gr}M$ over $\mathbb{C}[z_1, \ldots, z_n]$.

Recall now that $\mathrm{gr}\mathsf{H}_{0,c}\mathsf{e} = SV$, $\mathrm{gr}\mathsf{Z}_{0,c} = (SV)^G$. Thus the $c = 0$ case implies the general case, and we are done.

Now let us show that any Azumaya point of M_c is smooth. Let U be an affine open set in M_c consisting of Azumaya points. Then the localization $\mathsf{H}_{0,c}(U) := \mathsf{H}_{0,c} \otimes_{\mathsf{Z}_{0,c}} \mathcal{O}_U$ is an Azumaya algebra. Moreover, for any $\chi \in U$, the unique irreducible representation of $\mathsf{H}_{0,c}(U)$ with central character χ is the regular representation of G (since this holds for generic χ by Proposition 8.24). This means that e is a rank 1 idempotent in $\mathsf{H}_{0,c}(U)/\langle\chi\rangle$ for all χ. In particular, $\mathsf{H}_{0,c}(U)\mathsf{e}$ is a vector bundle on U. Thus the functor $F : \mathcal{O}_U\text{-mod} \to \mathsf{H}_{0,c}(U)\text{-mod}$ given by the formula $F(Y) = \mathsf{H}_{0,c}(U)\mathsf{e} \otimes_{\mathcal{O}_U} Y$ is an equivalence of categories (the quasi-inverse functor is given by the formula $F^{-1}(N) = \mathsf{e}N$). Thus $\mathsf{H}_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, and therefore their homological dimensions are the same.

On the other hand, the homological dimension of $\mathsf{H}_{0,c}$ is finite (in fact, it equals to $\dim V$). To show this, note that by the Hilbert syzygies theorem, the homological dimension of SV is $\dim V$. Hence, so is the homological dimension of $G \ltimes SV$ (as $\mathrm{Ext}^*_{G\ltimes SV}(M, N) = \mathrm{Ext}^*_{SV}(M, N)^G$). Thus, since $\mathrm{gr}\mathsf{H}_{0,c} = G \ltimes SV$, we get that $\mathsf{H}_{0,c}$ has homological dimension $\le \dim V$. Hence, the homological dimension of $\mathsf{H}_{0,c}(U)$ is also $\le \dim V$ (as the homological dimension clearly does not increase under the localization). But $\mathsf{H}_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, so $\mathcal{O}_U$ has a finite homological dimension. By Theorem 8.43, this implies that U consists of smooth points.

Corollary 8.44. $\mathrm{Az}(\mathsf{M}_c)$ *is also the set of points at which the Poisson structure of* M_c *is symplectic.*

Proof. The variety M_c is symplectic outside of a subset of codimension 2, because so is M_0. Thus the set S of smooth points of M_c where the top exterior power of the Poisson bivector vanishes is of codimension ≥ 2. Since the top exterior power of the Poisson bivector is locally a regular function, this implies that S is empty. Thus, every smooth point is symplectic, and the corollary follows from the theorem. □

8.13. **Notes.** Our exposition in this section follows Section 8 – Section 10 of [E4].

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