18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 10

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Recall that we had left to show that there are no surfaces in characteristic $p > 0$ satisfying

(1) $\mathrm{Pic}\,(X)$ is generated by $\omega_X = \mathcal{O}_X(K)$, and the anticanonical bundle is ample. In particular, X doesn't have any nonsingular rational curves.
(2) Every divisor of $|-K|$ is an integral curve of arithmetic genus 1.
(3) $(K^2) \leq 5, b_2 \geq 5$.

Lemma 1. *Let X be as above. Then $\exists$ a nonsingular curve $D \in |-K|$.*

Proof. Suppose not. Since $p_a(D) = 1$, we may assume that every D in $|-K|$ has exactly one singular point, a node or a cusp. Then $\dim |-K| \geq K^2 \geq 1$. Let $L \subset |-K|$ be a one-dimensional linear subsystem. The fiber s of $\phi_L : X \dashrightarrow \mathbb{P}^1$ are exactly the curves in L, because L has no fixed components (all the elements of L are integral). Let Y be the set of all singular points on curves in L, and let x be a base point of L (if any). We claim that $x \notin Y$. This is because if $x \in Y$, then for $D \in L$ a curve singular at x, $\pi : \tilde{X} \to X$ the blowup of X at x, then $\tilde{D}$ is a nonsingular rational curve and a fiber of $\tilde{\phi} = \phi \circ \pi : \tilde{X} \to \mathbb{P}^1$. After further (at most K^2) blowups, we get an X' s.t. $\phi' : X' \to \mathbb{P}^1$ is a morphism and one fiber of ϕ' is a smooth, rational curve. So X' is geometrically ruled over $\mathbb{P}^1$, implying that X' is rational and so X is rational. This, however, is impossible by the classification of rational surfaces ($\mathrm{Pic}\,(X)$ is never $\mathbb{Z}\omega_X$), giving the desired contradiction.

So blowup the base points of ϕ so that $\tilde{\phi} : \tilde{X} \to \mathbb{P}^1$ is a quasi-elliptic fibration (by Tate, all singular points are cusps in characteristic 2 or 3, e.g. $y^2 = x^3 + t$). All the fibers are integral rational curves with one singular point, and the set of singularities $\tilde{Y}$ of $\tilde{\phi}$ is a nonsingular irreducible curve with $\tilde{\phi}|_Y : \tilde{Y} \to \mathbb{P}^1$ a bijective, purely inseparable morphism. Thus, $\tilde{Y} \cong \mathbb{P}^1 \implies Y \cong \mathbb{P}^1$, contradicting the fact that there are no smooth rational curves on X. $\square$

Lemma 2. *Let X be as above. Then $H^2(X, T_X) = 0$ for $T_X = (\Omega^1_{X/k})^\vee$ the tangent sheaf.*

Proof. $\exists$ a nonsingular elliptic curve $D \in |-K|$ by the above lemma. The short exact sequence

$$0 \to \mathcal{O}_X((n-1)D) \otimes T_X \to \mathcal{O}_X(nD) \otimes T_X \to \mathcal{O}_X(nD) \otimes T_X \otimes \mathcal{O}_D \to 0 \tag{1}$$

gives the long exact sequence in cohomology

$$\begin{aligned} H^1(X, \mathcal{O}_X(nD) \otimes T_X \otimes \mathcal{O}_D) \to H^2(X, \mathcal{O}_X((n-1)D) \otimes T_X) \\ \to H^2(D, \mathcal{O}_X(nD) \otimes T_X) = 0 \end{aligned} \tag{2}$$

for large n by Serre vanishing (since D is ample). By reverse induction, it is enough to show that $H^1(X, \mathcal{O}_X(nD) \otimes T_X \otimes \mathcal{O}_D) = 0$ for $n \geq 1$. This will show

$$\begin{aligned} H^2(\mathcal{O}_X((n-1)D) \otimes T_X) = 0 \implies H^2(\mathcal{O}_X((n-2)D) \otimes T_X) = 0 \\ \implies H^2(\mathcal{O}_X \otimes T_X) = 0 \end{aligned} \tag{3}$$

Dualizing the conormal exact sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_D \to \Omega_D \to 0$, we get

$$0 \to \mathcal{O}_D \cong T_D \to T_X \otimes \mathcal{O}_D \to N = \mathcal{O}_X(D) \otimes \mathcal{O}_D \to 0 \tag{4}$$

where we used that $T_D = \omega_D^\vee \cong \mathcal{O}_D$ since D is an elliptic curve. Taking cohomology gives

$$H^1(\mathcal{O}_X(nD) \otimes \mathcal{O}_D) \to H^1(\mathcal{O}_X(nD) \otimes T_X \otimes \mathcal{O}_D) \to H^1(\mathcal{O}_X(nD) \otimes N) \tag{5}$$

$D^2 > 0$, so we get $H^1(\mathcal{O}_X(nD) \otimes \mathcal{O}_D) = 0 = H^1(\mathcal{O}_X(nD) \otimes N)$ as desired, since $\mathcal{O}_X(nD) \otimes \mathcal{O}_D$ and $\mathcal{O}_X(nD) \otimes N$ are sheaves of large positive degree on D for $n >> 0$. $\square$

We now return to the proof of the proposition for $p = \mathrm{char}(k) > 0$. Let $A = W(k)$ be the Witt vectors of k. A is a complete DVR of characteristic 0, with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} = k$. The idea is to lift X to characteristic 0 and use the result already proved. Note that $H^2(X, T_X) = 0$: in addition, X is projective and $H^2(X, \mathcal{O}_X) = 0$, so by SGA1, Theorem III, 7.3, there is a smooth projective morphism $f : U \to V = \mathrm{Spec}\,(A)$ which closed fiber isomorphic to X. Let X' be the general fiber. Then X' is a nonsingular projective surface over the fraction field K' of A (which is unfortunately not algebraically closed). The fibers of f are 2-dimensional, so $R^i f_* \mathcal{O}_U = 0$ for $i \geq 3$. The base change theorem gives

$$(R^2 f_* \mathcal{O}_U) \otimes_A A/\mathfrak{m} \to H^2(f^{-1}(\mathfrak{m}), \mathcal{O}_U/\mathfrak{m}\mathcal{O}_U) = H^2(X, \mathcal{O}_X) = 0 \tag{6}$$

is an isomorphism. By Nakayama's lemma, we get that $R^2 f_* \mathcal{O}_U = 0$, and similarly for R^1. Thus, $H^1(X', \mathcal{O}_{X'}) = H^2(X', \mathcal{O}_{X'}) = 0$. See Mumford's Abelian Varieties or Chapters on Algebraic Surfaces for more details.

Now, let K'' be an algebraic closure of K' and K_i' the family of finite extensions of K' inside K''. Let $X'' = X' \times_{K'} k''$, $X_i = X' \times_{K'} K_i'$. Let A'' be the integral closure of A inside K'', and $A_i = A'' \cap K_i''$. Let $\mathfrak{m}''$ be a maximal ideal of A'' lying over $\mathfrak{m}$ and $B'' = A''_{\mathfrak{m}''}$ its localization. Similarly, let $\mathfrak{m}_i = \mathfrak{m}'' \cap A_i, B_i = (A_i)_{m_i}$ and set $\mathfrak{n}'' = \mathfrak{m}''B'', \mathfrak{n}_i = \mathfrak{m}_i B_i$. Since $K = A/\mathfrak{m}$ is algebraically closed, we see, $B''/\mathfrak{n}'' = B_i/\mathfrak{n}_i = K$.

Now let $V_i = \operatorname{Spec} B_i, U_i = U \otimes_A B_i, f_i = f \otimes_A B_i : U_i \to V_i$.

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & U_i \arrow[r] \arrow[d, "f_i"] & U \arrow[d, "f"] & X \arrow[l] \arrow[d] \\ \operatorname{Spec} K_i' \arrow[r] & V_i \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} K \arrow[l] \end{tikzcd} \tag{7}$$

Since $B_i/\mathfrak{n}_i = k$, the closed fiber of f_i is canonically isomorphic to X. The generic fiber of f_i is isomorphic to X_i and since $K_i \supset K$ is finite, B_i is a DVR and $\{V_i\}$ is an inductive system. By EGA and general nonsense, $\varinjlim \text{Pic } X_i \to \text{Pic } X''$ is an isomorphism.

Lemma 3. *There is a group isomorphism $b : \text{Pic } X_i \to \text{Pic } X$ defined by the following: for L an invertible $\mathcal{O}_{X_i}$-module, $\exists$ an invertible $\mathcal{O}_{U_i}$-module L_i s.t. $L_i|_{X_i} \cong L$, and we set $b([L]) = [L_i|_X]$.*

Proof. Omitted. ☐

Proof of theorem: So we get a canonical isomorphism between Pic X and Pic X'' which takes ω_X to $\omega_{X''}$. Since $\operatorname{Pic} X = \mathbb{Z}\omega_X$, we see that $\text{Pic } X'' = \mathbb{Z}\omega_{X''}$ and $\omega_{X''}^{-1}$ is ample, giving us (1) for X''. Also, $\omega_{X''} \cdot \omega_{X''} = \omega_{X'} \cdot \omega_{X'} = \omega_X \cdot \omega_X$ using flatness and the definition of $(\cdot)$ by $\chi()$, so $\omega_{X''} \cdot \omega_{X''} \leq 5$ and $b_2(X'') \geq 5$ by Noether's formula. Since $q(X') = 0, q(X'') = 0$. But X'' is over an algebraically closed field of characteristic 0, which as shown last time is impossible. ☐

Definition 1. *A surface X is* unirational *if there is a dominant morphism $Y \to X$ from a rational surface.*

Corollary 1. *In characteristic zero, a unirational surface is rational.*

Note that this is not true in characteristic > 0, e.g. the Zariski surfaces $z^p = f(x, y)$.

Proof. Given $f : Y \to X$ where Y is rational, we have $q(Y) = p_2(Y) = 0$. f is separable, so it induces an injective map $H^0(X, \omega_X^{\otimes n}) \to H^0(Y, \omega_Y^{\otimes n})$. Thus, $p_2(X) = 0$ and similarly $q(X) = 0$. By the above, it is rational. ☐

Remark. In Castelnuovo's theorem, it is not enough to take $q = p_g = 0$: counterexamples include the Godeaux surfaces, e.g. a quotient of $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{P}^3$ by $\mathbb{Z}/5\mathbb{Z}$ acting as $[w : x : y : z] \mapsto [w : \zeta_5 x : \zeta_5^2 y : \zeta_5^3 z]$, and Enriques surfaces (quotients of $K3$ surfaces by fixed point free involutions).

1. Picard and Albanese Varieties

Let X be a smooth variety, $\operatorname{Pic} X$ the Picard group of line bundles up to isomorphism (or divisors up to linear equivalence). Set $\operatorname{Pic}^0(X)$ to be those algebraically equivalent to zero: we say that L_1 and L_2 are algebraically equivalent (i.e. $L_1 \approx L_2$) if $\exists$ a connected scheme T, points $t_1, t_2 \in T$, and an invertible $\mathcal{O}_{X\times T}$-module L s.t. $L|_{X\times\{t_1\}} \cong L_1$ and $L_{X\times\{t_2\}} \cong L_2$. For such L_1, L_2, $L_1 \otimes L_2^{-1}$ is algebraically equivalent to zero. The Picard functor from schemes over k to sets is that which maps a scheme T to the set $\operatorname{Pic}_X(T)$ of all T-isomorphism classes of invertible $\mathcal{O}_{X\times T}$-modules, where L_1, L_2 are T-isomorphic if $\exists$ and invertible $\mathcal{O}_T$-module M s.t. $L_1 \cong L_2 \otimes p_2^*(M)$. This is a contravariant functor: for $f : T' \to T$ a morphism of k-schemes, then

$$\operatorname{Pic}_X(f) : \operatorname{Pic}_X(T) \to \operatorname{Pic}_X(T'), \operatorname{Pic}_X(f)([L]) = [(id_X \times f)^*[L]] \tag{8}$$

Pic_X has a subfunctor Pic^0_X defined by

$$\operatorname{Pic}^0_X(T) = \{L \in \operatorname{Pic}_X(T) | L_t = L|_{X\times\{t\}} \sim_{alg} \mathcal{O}_X \text{ for all } t \in T\} \tag{9}$$

Theorem 1. *The functors* $\operatorname{Pic}_X, \operatorname{Pic}^0_X$ *are representable by schemes called* $\underline{\operatorname{Pic}}_X$ *and* $\underline{\operatorname{Pic}}^0_X$ *respectively.*

This means that we have a natural equivalence $\operatorname{Pic}_X(T) = \operatorname{Hom}_{k-Sch}(T, \underline{\operatorname{Pic}}_X)$, and $T \to \underline{\operatorname{Pic}}_X$ corresponds uniquely to a line bundle on $X \times T$ up to T-isomorphism. The identity map $\underline{\operatorname{Pic}}_X \to \underline{\operatorname{Pic}}_X$ corresponds to a line bundle $\mathcal{L}$ on $X \times \underline{\operatorname{Pic}}_X$ called the *universal bundle*: the map $f : T \to \underline{Pic}_X$ corresponds to the line bundle $(\mathrm{id}_X \times f)^*(\mathcal{L})$ on $X \times T$. Similarly for $\underline{Pic}^0_X$.