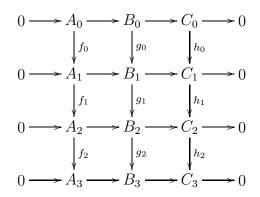
18.726 Algebraic Geometry Spring 2009

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## 18.726: Algebraic Geometry (K.S. Kedlaya, MIT, Spring 2009) Problem Set 2 (due Friday, February 20, in class)

Please submit exactly *twelve* of the following exercises, including all exercises marked "Required".

- 1. (Required) Prove that <u>Set</u> and its opposite category are not equivalent, by finding an arrow-theoretic property satisfied by <u>Set</u> but not by its opposite, or vice versa. (Many solutions are possible.)
- 2. Prove that the sheafification functor from presheaves on a topological space X to sheaves on X, and the forgetful functor from sheaves to presheaves, form an adjoint pair.
- 3. (Required)
  - (a) Complete the proof of the basis lemma ("sheaves" handout) in the case of a nice basis.
  - (b) Complete the proof of the basis lemma in general.
- 4. Let  $i: Z \hookrightarrow X$  be an inclusion of topological spaces in which Z carries the subspace topology. Let  $\mathcal{F}$  be a sheaf on X.
  - (a) If Z is open, prove that  $i^{-1}\mathcal{F}$  may be canonically identified with the restriction  $\mathcal{F}|_Z$ .
  - (b) If  $Z = \{x\}$ , prove that  $i^{-1}\mathcal{F}$  may be canonically identified with the stalk  $\mathcal{F}_x$ .
- 5. (Required if you've never done it before) Prove the five lemma ("abelian sheaves" handout).
- 6. (Required if you've never done it before)
  - (a) Complete the proof of the snake lemma ("abelian sheaves" handout).
  - (b) Let



be a commutative diagram in which the rows are exact and the columns are complexes. Use the snake lemma to show that

$$\frac{\ker(f_1)}{\operatorname{im}(f_0)} \to \frac{\ker(g_1)}{\operatorname{im}(g_0)} \to \frac{\ker(h_1)}{\operatorname{im}(h_0)} \xrightarrow{\delta} \frac{\ker(f_2)}{\operatorname{im}(f_1)} \to \frac{\ker(g_2)}{\operatorname{im}(g_1)} \to \frac{\ker(h_2)}{\operatorname{im}(h_1)}$$

is exact, where  $\delta$  is defined as in the snake lemma, and the other maps are induced naturally by the horizontal arrows. (This will later give us the *long exact sequence* in homology.)

- 7. (a) Let  $f^*: \mathcal{C}_1 \to \mathcal{C}_2$  and  $f_*: \mathcal{C}_2 \to \mathcal{C}_2$  be an adjoint pair of functors, where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are abelian categories. Prove that  $f^*$  is right exact and  $f_*$  is left exact.
  - (b) Use this to prove that as functors from  $\underline{\mathrm{Mod}}_R$  to itself (for any given ring R),  $\mathrm{Hom}(X,\cdot)$  is left exact and  $X\otimes_R\cdot$  is right exact. (I.e., prove that these two form an adjoint pair.)
- 8. The Taylor series map gives a ring homomorphism from the germs at 0 of the sheaf of holomorphic functions on  $\mathbb{C}^n$  to  $\mathbb{C}[x_1,\ldots,x_n]$ . Prove that this map is injective, and that its image consists of the power series which converge on some open polydisc around 0.
- 9. Do Hartshorne II.1.3, but for (b), instead of the example given in class, make an example on the space  $\{1, 2, 3\}$  with topology generated by  $\{1, 2\}, \{2, 3\}$ .
- 10. (Required)
  - (a) Prove Hartshorne II.1.7 by checking that formation of kernel, image, and cokernel of a morphism of abelian sheaves commute with passage to stalks.
  - (b) Prove that abelian sheaves on a fixed topological space form an abelian category in which the categorical kernel, image, and cokernel coincide with the notions we defined.
- 11. Prove that a preadditive category with finite products is additive, that is, the product can be naturally equipped with a biproduct structure.
- 12. A topological space is *quasicompact* if every open cover admits a finite subcover. (The term *compact* is usually reserved for a space which is not just quasicompact but also Hausdorff.)
  - (a) Recall that for any ring R, Spec(R) is quasicompact. (We did this in class, so you don't have to write anything here.)
  - (b) Prove that if R is a noetherian ring, then any subset of Spec(R) is quasicompact.
  - (c) Prove that (b) can fail if R is not noetherian. (Hint: construct a ring in which  $\operatorname{Spec}(R)$  contains an infinite subset carrying the discrete topology.)

- 13. (Required) Describe Spec  $\mathbb{Z}$  by listing:
  - (a) the points;
  - (b) the open sets;
  - (c) the sections of the structure sheaf over each open set.
- 14. (a) What is the maximum number of elements of  $\mathbb{Z}$  which do not have the same value at any point of Spec( $\mathbb{Z}$ )?
  - (b) Same question with  $\mathbb{Z}$  replaced by the ring  $\mathbb{Z}[i]$  of Gaussian integers?
- 15. (Required) Hartshorne II.2.1 and II.2.2 (these count as one exercise).
- 16. This exercise is an arithmetic analogue of the fact that the complement of the origin in  $\mathbb{A}^2$  is not an affine algebraic variety. Let X be the locally ringed space obtained from Spec  $\mathbb{Z}[x]$  by removing the point (2, x).
  - (a) Prove that  $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}[x]$ . (Hint: cover X with the distinguished opens D(2) and D(x) of Spec  $\mathbb{Z}[x]$ .)
  - (b) Use (a) to show that X is not affine.
- 17. Hartshorne II.2.10 or Hartshorne II.2.11 (but not both).