Lecture 8: Product Topology, Complete Varieties

To check that $\mathbb{P}^n$ is separated, we used an affine covering of $\mathbb{P}^n$ as $\cup\mathbb{A}^n_i$. Instead, we could have checked that the preimage of $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ in $(\mathbb{A}^{n+1} \setminus 0)^2$ is closed; this is given by the equation $X \wedge Y = 0$ (recall that $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/k^\times$.

Remark 1. *We have that X is Hausdorff if and only if the diagonal in X^2 is closed with respect to the **product topology**, and **not** the Zariski topology.*

Corollary 1. *If $k = \mathbb{C}$, then X is separated iff and only if X_{cl} (which is X with the classical topology coming from $\mathbb{C}$) is Hausdorff.*

Proof. Let X be a variety over k, and $Z \subseteq X$ be a Zariski locally closed subset. We claim that Z is Zariski closed if and only if it is classically closed. To see this, it suffices to check that if Z is Zariski locally closed and classically closed, then it is Zariski closed. Note that Z is Zariski open in $\overline{Z_{\text{Zar}}}$, and so it is open dense in $\overline{Z_{\text{cl}}}$, so $\overline{Z_{\text{Zar}}} = \overline{Z_{\text{cl}}}$. Since the diagonal Δ is Zariski locally closed, we are done. ☐

Remark 2. *The image of a morphism may not be a subvariety. For example, take the map from $\mathbb{A}^2$ to itself induced by the polynomial mapping $k[a,b] \to k[x,y]$, $a \mapsto x, b \mapsto xy$. The image is $\{a \neq 0\} \cup \{(0,0)\}$. It is not a subvariety, but it will be a constructible subset (this is Chevalley's Theorem, which will be proven later). Suppose X, Y are irreducible and $f : X \to Y$ is a morphism. Then either $f(X)$ is contained in a closed subset $Z \supsetneq Y$, or $f(X)$ contains an open dense subset U.*

Proposition 1. *X is separated if and only if for any affine open $U, V \subseteq X$, $U \cap V$ is affine and $k[U \cap V]$ is generated by $k[U]$ and $k[V]$.*

Proof. Consider an open $U \times V \subseteq X \times X$ where U, V are open subsets in X. Since X is separated, the intersection of diagonal with $U \times V$ is closed in $U \times V$; furthermore, this intersection equals $U \cap V$. As $U \times V$ is affine and $U \cap V$ is closed, we see that $U \cap V$ is affine. We also have $k[U] \otimes k[V] = k[U \times V] \twoheadrightarrow k[U \cap V]$.

For the converse, the second condition implies that $(U \times V) \cap \Delta$ is closed in $U \times V$, so Δ is closed. ☐

Example 1. *Let X be the affine line with a doubled origin, with the usual affine open covering $U \cup V$ where $U = \mathbb{A}^1_1$, $V = \mathbb{A}^1_2$. Then this covering corresponds to $k[t_1, t_2] \mapsto k[t, t^{-1}]$ where $t_1, t_2 \mapsto t$. This is not surjective.*

Consider X to now be the affine plane with a doubled origin, with affine open covering $U \cup V$ where $U = \mathbb{A}^2_1$, $V = \mathbb{A}^2_2$. In this case, $U \cap V = \mathbb{A}^2 \setminus \{0\}$ is not affine.

Also, we checked last time that for Y separated, $f : X \to Y$ is determined by $f|_U$ where U is a dense open subset of X.

Proposition 2. *(Caternary property). Let X be an algebraic variety, with $X = Z_n \supsetneq Z_{n-1} \supsetneq \cdots \supsetneq Z_0$ where each Z_i is closed irreducible. If this chain cannot be refined, then $\dim Z_i = i$.*

Proof. Theorem 2.6.7 of [K]. ☐

Now we consider "dimension and rate of growth." Let A be a finitely generated k-algebra. Let V be the space of generators. Set $V_n = \operatorname{span}\{x_1 \cdots x_k : x_i \in V, k \leq n\}$ and $D_V(n) = \dim V_n$. The asymptotic behavior of $D_V(n)$ actually does not depend on V. For if $V' \subseteq V_d$, then $D_{V'}(n) \leq D_V(nd)$.

Proposition 3. *If $A = k[X]$ where X is affine of dimension d, then $D_V(n) = \Theta(n^d)$; that is, there exist constants c', c such that for all n,*

$$c'n^d \leq D_V(n) \leq cn^d \tag{*}$$

Proof. Suppose $B \subseteq A$ and A is finite over B. If (*) holds for B, then it holds for A. Given V_B to be generators for B, $V_A = V_B \cup W$ where W are generators for A as a B-module, note each $x \in W$ satisfies an equation of the form $x^r = b_{r-1}x^{r-1} + \cdots + b_0$ for $b_i \in B$. We can assume without loss of generality that $b_i \in V_B$. Then $D_{V_B}(n) \leq D_{V_A}(n) \leq D_{V_B}(n) \cdot c$ where $c = r^{\dim W}$. Setting $B = k[x_1, \cdots, x_d]$, an explicit computation gives a polynomial in n of degree d. ☐

Remark 3.

(1) *The order of growth function has been used to generalized the concept of dimension to noncommutative algebras, groups etc. in the works of Artin, Gromov and others.*

(2) *In our commutative setting the function $D_V(n)$ can in fact be analyzed much more precisely. It turns out that for large n we have $D_V(n) = P(n)$ for a certain polynomial P. It is closely related to the so called Hilbert polynomial, to be described in 18.726.*

Theorem 1.1. *Suppose X, Y are irreducible subvarieties in $\mathbb{A}^n$. Then each component of $X \cap Y$ has codimension at most* $\operatorname{codim} X + \operatorname{codim} Y$.

Proof. Rewrite $X \cap Y = (X \times Y) \cap \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{A}^n$. From last time, $\dim(X \times Y) = \dim X + \dim Y$. The diagonal in affine space is cut out by the n linear equations $x_i = y_i$. By a theorem of last time we know that each component of $Z_f \subseteq X$ has dimension equal to $\dim X - 1$, so $\dim(X \cap Y) \geq \dim(X \times Y) - n = \dim X + \dim Y - n$. □

Remark 4. *This theorem doesn't exclude empty intersections. The obvious example is the intersection of subvarieties $x_1 = 0$ and $x_1 = 1$.*

Theorem 1.2. *The previous theorem holds for $X, Y \subseteq \mathbb{P}^n$; moreover, the intesection $X \cap Y$ is nonempty if $\dim X + \dim Y > n$.*

Proof. Here is a lemma: the dimension of C_X (the cone over X) equals $\dim X + 1$. To see this, note that $C_X \cap \{x_i = 1\}$ is isomorphic to $U_i = X \cap \mathbb{A}^n_i \subseteq X$, and from this it is a straightforward exercise to complete the proof of this lemma.

Using this, the proof of the theorem goes as follows: $\dim(X \cap Y) = \dim C_{X \cap Y} - 1 = \dim(C_X \cap C_Y) - 1 \geq \dim C_X + \dim C_Y - (n+1) - 1 = \dim X + \dim Y - n$. The intersection of cones is nonempty as it contains 0. □

Complete varieties

Definition 1. *A variety X is complete if it is separated and universally closed, which means that for all Y, the projection map $Y \times X \to Y$ sends closed sets to closed sets.*

We will see that for $k = \mathbb{C}$, X is complete if and only if X_{cl} is compact. Also, if X is quasiprojective, we will see that complete is equivalent to projective. For the forward direction, suppose $\iota : X \hookrightarrow \mathbb{P}^n$ is locally closed. Then X is in the image of the closed embedding $\Gamma_\iota \hookrightarrow X \times \mathbb{P}^n$, so $X \subseteq \mathbb{P}^n$ is closed.

Lemma 1.

(i) *Suppose Z is closed in X. Then X is complete implies Z is complete.*

(ii) *If $f : X \to Z$ is a morphism with Z separated and X complete, then $f(X) \subseteq Z$ is a closed complete subvariety.*

(iii) *If X, Y are complete, then so is $X \times Y$.*

Proof. (i) We see that $Y \times Z$ is closed in $Y \times X$, so by considering the projection to Y, this is clear.

(ii) Identify $f(X)$ with Γ_f in $X \times Z$. As X, Z are separated, so is $X \times Z$. As Γ_f is a closed subvariety of $X \times Z$, it is also separated (for these facts, see Lemma 3.3.2 of [K]). Hence $f(x)$ is separated.

To check that $f(X)$ is universally closed, take a variety Y and closed subset $T \subseteq f(X) \times Y$. It suffices to check that the image of T in Y is closed. Consider the map $f \times \mathrm{id} : X \times Y \to f(X) \times Y$, and let $\widetilde{T} = (f \times \mathrm{id})^{-1}(T) \subseteq X \times Y$. Then it suffices to check that the image of $\widetilde{T}$ under the projection $X \times Y \to Y$ is closed, which follows from X being complete.

(iii) As X, Y are both separated, so is $X \times Y$ (Lemma 3.3.2 of [K]).

Let Z be any variety and $T \subseteq X \times Y \times Z$ closed. As X is universally closed, the image of T in $Y \times Z$ is closed. As Y is universally closed, the image of T in Z is closed. Hence, $X \times Y$ is universally closed. $\square$

Proposition 4. $\mathbb{P}^n$ *is complete.*

Proof. We know $\mathbb{P}^n$ is separated (Lemma 3.3.2 of [K]), so it suffices to check that it is universally closed.

We use an "elimination theory" argument. Let Y be any variety and $Z \subseteq \mathbb{P}^n \times Y$ be a closed subset. Then Z comes from a closed subset $\widetilde{Z} \subseteq \mathbb{A}^{n+1} \times Y$. Suppose $I_{\widetilde{Z}}$, the ideal of functions vanishing on $\widetilde{Z}$, is generated by some homogeneous polynomials $P_i \in k[Y][x_0, \cdots, x_n]$. For $y \in Y$, let $P_{i,y} = P_i(y, -) \in k[x_0, \cdots, x_n]_d$ for some d (this is the degree d homogeneous polynomials). Then $(P_{i,y})$ is an ideal of $k[x_0, \cdots, x_n]$, so we let $U_d = \{y \in Y : (P_{i,y}) \supseteq k[x_0, \cdots, x_n]_d\}$. Letting $\mathrm{pr}(Z)$ be the image of Z in $\mathbb{P}^n \times Y \to Y$, we see that $y \notin \mathrm{pr}(Z)$ iff there is no point $(x_0, \cdots, x_n)$ which makes all of the $P_{i,y}$ vanish, iff it lies in some U_d. Therefore, $Y \setminus \mathrm{pr}(Z) = \cup_d U_d$. It is enough to check that each U_d is open, which is equivalent to checking that the natural map $\oplus_i k[x_0, \cdots, x_n]_{d-d_i} \to k[x_0, \cdots, x_n]_d$ (where d_i is the degree of P_i) defined by sending $(g_i) \mapsto \sum g_i P_{i,y}$ is surjective. This is equivalent to requiring that some matrix with $k[Y]$-entries, when evaluated at y, has maximal rank, which is some condition of non-vanishing of minors. So it is an open condition. $\square$

So projective varieties are complete, and a quasiprojective variety is complete if and only if it is projective.