1 Recap of last time

Def. An *alg. subset* of k^n or a subset which is *algebraically closed* is a set of the form $V(I) = \{\underline{a} \in k^n | f(\underline{a}) = 0 \forall f \in I\}$ where $I \subset k[\underline{x}]$ is an ideal.

Thm. 1 There is a natural bijection between

$$\{\text{points in } V(I)\} \leftrightarrow \{\text{max. ideals in } k[\underline{x}]/I\}.$$

I and $\sqrt{I}$ define the same algebraic set. (If we consider $k[\underline{x}]/I$ and $k[\underline{x}]/\sqrt{I}$ the maximal ideals are the same; the only difference is the possible presence of some nilpotent elements.

Thm 2. There is a natural bijection

$$\{\text{alg. subsets of } k^n\} \leftrightarrow \{\text{ideals } I \subset k[x_1, \dots, x_n] \text{ with } I = \sqrt{I}.$$

In the forward direction, this is $I : \Sigma \mapsto \{f \in k[x_1, \dots, x_n] || f(\Sigma) = 0\}$ and in the reverse direction this is $V : A \mapsto \{\underline{a} \in k^n | f(\underline{a} = 0 \forall f \in A\}$.

Proof.

It should be clear that $V(I(\Sigma)) \supset \Sigma$. If $\underline{a} \notin \Sigma$ then there is an $f \in A$ such that $f(\underline{a}) \neq 0$. Thus, $V(I(\Sigma) \subset \Sigma$.

Note: $I(\Sigma) = \sqrt{I(\Sigma)}$. By definition $\sqrt{I(\Sigma)}$ is the set of functions where some power of that function vanishes on Σ, but this is over a field, so the function itself vanishes on Σ.

Thm 2.1 (sometimes called Nullstellensatz.) Let $A = (f_1, \dots, f_m)$, and let $g \in k[x_1, \dots, x_n]$ such that if $f_1(\underline{a}) = \dots = f_m(\underline{a}) = 0$ then $g(\underline{a}) = 0$. Then we must show that $g^l = \sum_i h_i f_i$ where $h_i \in k[x_1, \dots, x_n]$ (ie, some power of g is a linear combination of fs).

Consider $k[x_1, \dots, x_{n+1}]$ and let $B = Ak[x_1, \dots, x_{n+1}] + (1 - gX_{n+1}k[x_1, \dots, x_{n+1}]$. Either B is a proper ideal or B is the whole ring.

If B is a proper ideal it lies in a maximal ideal $m = (x_1 - a_1, \dots, x_{n+1} - a_{n+1})$ since any maximal ideal is of this form. Thus, $f_1(\underline{a}) = \dots = f_m(\underline{a})$ so $g(a_1, \dots, a_n) = 0$ and $1 = g(a_1, \dots, a_n)a_{n+1} = 0$; contradiction.

If B is the whole ring $k[x_1, \dots, x_{n+1}]$ so $1 \in B$, so $\sum_i h_i^*(x_1, \dots, x_{n+1})f_i + h_{r+1}^*(x_1, \dots, x_{n+1})(1 - gX_{n+1})$. So, we get

$$1 = \sum_i h_i^*(x_1, \dots, x_n, g^{-1})f_i(\underline{x}),$$

in $k[x_1, \dots, x_n]_g$, but this is not a problem since $k[x_1, \dots, x_n]$ injects into this ring.

The h_i^*s are sums of monomials of the form $x_1^{\alpha_1} \dots x_n^{\alpha_n} g^{-\alpha_{n-1}}$. We choose l large enough to clear the g^{-1} terms from all these monomials, and we get

$$g^l = \sum_i h_i(x_1, \dots, x_n)f_i(\underline{x}),$$

since now we can rewrite h_i as some sum of monomials in $x_1, \dots, x_n$ multiplied by g, which is also a polynomial in $x_1, \dots, x_n$.

This proves theorem 2.1 which proves theorem 2.

2 Irreducible components

Lemma 3 i. $A \subset B \Rightarrow V(A) \supset V(B)$.
ii. $\Sigma_1 \subset \Sigma_2 \Rightarrow I(\Sigma_1) \supset I(\Sigma_2)$.
iii. $V(\sum_\alpha A_\alpha = \cap_\alpha V(A_\alpha)$.
iv. $V(A \cap B) = V(A) \cup V(B)$.

Pf. (i - iii) are pretty easy. For iv., it is easy to see that $V(A) \cup V(B) \subset V(A \cap B)$,
Conversely, if $x \notin V(A) \cup V(B)$ then there is some $f \in A$ and $g \in B$ such that $f(\underline{x}) \neq 0$ and $g(\underline{x}) \neq 0$. Then, $fg \in A \cap B$ but $fg(\underline{x}) \neq 0$ so $\underline{x} \notin V(A \cap B)$.

Def. An algebraic subset $\Sigma \subset k^n$ is *irreducible* if it cannot be written as a union of algebraic subsets $\Sigma = \Sigma_1 \cup \Sigma_2$ with neither $\Sigma_i = \Sigma$.

Thm. 4 V and I induce a bijection between irreducible algebraic subsets of k^n and prime ideals in $k[x_1, \ldots, x_n]$.
(Note: if P is a prime ideal, then $P = \sqrt{P}$, since A/P is an integral domain, and integral domains have no nilpotent elements.)

Lemma 5. If A is prime, then $V(A)$ is irreducible.

Pf. of lemma 5. Suppose not. Suppose $V(A) = V(A_1) \cup V(A_2)$ where each is a proper subset of $V(A)$. $A \subsetneq A_i$ for each i. In fact, $A = A_1 \cap A_2$. Now let $x \in A_1 - A$ and $y \in A_2 - A$. Then $xy \in A_1 \cap A_2 = A$ but neither x nor y in A, which contradicts that A is prime.
Conversely, we can use the primary decomposition theorem to prove that if A is not prime then $V(A)$ is not irreducible.

Thm 6. (Primary Decomposition) Any radical ideal can be written uniquely as an intersection of prime ideals. (See [AM].)

Pf. of theorem 4. One direction is proved by lemma 5. Given Σ we can write $I(\Sigma) = \cap_j P_j$ where P_j is prime. But then, if $I(\Sigma)$ is not prime, then there are at least two primes in this intersection, so $I(\Sigma) = P_i \cap (\cap_{j \neq i} P_j)$ but then $\Sigma = V(P_i) \cup V(\cap_{j \neq i} P_j)$ so Σ is not irreducible since $A \supsetneq P_i \supsetneq I(\Sigma)$ This completes the proof.

Example. Describe the irreducible components of $\Sigma \subset k^3$ defined by $A = (x^2 - yz, xz - x)$.
From a geometric point of view. Claim: $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ $xz - x = 0$ so either $x = 0$ or $z = 1$. If $x = 0$ then $yz = 0$ so either $y = 0$ or $z = 0$. If $z = 1$ then $x^2 = y$, so these are the three components: $(0, 0, z), (0, y, 0), (x, x^2, 1)$.
$(0, 0, z) = V(P_1 = (x, y))$, $(0, y, 0) = V(P_2 = (x, z))$, and $(x, x^2, 1) = V(P_3 = (z - 1, y - x^2))$. We need that these three ideals are prime, or that $k[x, y, z]/P_i$ is an integral domain. Each of the first two is just a polynomial ring in one variable; the third gives the same in a more complicated way.

Thus, $\sqrt{A} = (x, y) \cap (x, z) \cap (z-1, y-x^2)$. In this case, $A = \sqrt{A}$ (prove this as an exercise.)

Example: (Twisted Cubic). $C \subset k^3$ given by $C = \{(a, a^2, a^3) | a \in k\}$. What is $I(C)$? How about $(xy - z)$? The quotient $k[x, y, z]/(xy - z) \cong k[x, y]$ shows this is the wrong dimension.

$I(C) = (z - x^3, y - x^2)$. There is an exact sequence

$$0 \to (z - x^3, y - x^2) \to k[x, y, z] \to Q \to 0,$$

where $Q = k[x, y, z]/(z - x^3, y - x^2)$ but $Q \cong k[x]$ where this isomorphism is defined by $x \mapsto x, y \mapsto x^2, z \mapsto x^3$.

3 Projective Space

Notation. $\mathbb{P}^n(k) = \{0 \neq (x_0, \ldots, x_n) \in k^{n+1}\}/(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n), \lambda \in k^\times$. Note: there is a set map $(x_0, \ldots, x_n) \mapsto (x_0 : x_1 : \ldots : x_n)$. Observation: if $f \in k[x_0, \ldots, x_n]$ is homogeneous, then it makes sense to say $f(x_0 : \ldots : x_n) = 0$. (Homogeneous means that every monomial has the same degree.) Thus, if we replace x_i with λx_i we get the whole result time λ^{n_0} where n_0 is the homogeneous degree.

Def. An *algebraic subset* of $\mathbb{P}^n(k)$ is a set of the form $f_1 = \ldots = f_r = 0$ for $f_i \in k[x_0, \ldots, x_n]$ which are each homogeneous.

Observation: $\mathbb{P}^n(k) = U_0 \cup \ldots \cup U_n$ where $U_i = \{(x_0 : \ldots : x_n | x_i \neq 0\}$. Thus, $U_i \cong k^n$ naturally, where we put a 1 in the ith position, and map other elements normally. If we have some $\Sigma \subset \mathbb{P}^n(k)$ an alg. subset, then $\Sigma \cap U_i \subset k^n$ is an algebraic set in the previous sense.

Define {alg. sets in $\mathbb{P}^n(k)$} $\overset{I}{\underset{V}{\leftrightarrow}}$ {hom. ideals in $k[x_0, \ldots, x_n]$}.

$I(\Sigma) = \{\text{hom. } f \in k[\underline{x}] | f(p) = 0 \forall p \in \Sigma\}$,

$V(A) = \{p \in \mathbb{P}^n(k) | f(p) = 0 \forall \text{hom. } f \in A\}$.

Thm. 7 V and I induce a bijection between algebraic sets in $\mathbb{P}^n(k)$ and homogeneous ideals $A \subset k[x_0, \ldots, x_n]$ which are radical except for $(x_0, \ldots, x_n)$ (this last being excluded because its algebraic set is empty in projective space.

Proof next time.