4 Representations of finite groups: further results

4.1 Frobenius-Schur indicator

Suppose that G is a finite group and V is an irreducible representation of G over $\mathbb{C}$. We say that V is

- of complex type, if $V \ncong V^*$,
- of real type, if V has a nondegenerate symmetric form invariant under G,
- of quaternionic type, if V has a nondegenerate skew form invariant under G.

Problem 4.1. *(a) Show that* $\mathrm{End}_{\mathbb{R}[G]} V$ *is* $\mathbb{C}$ *for* V *of complex type,* $\mathrm{Mat}_2(\mathbb{R})$ *for* V *of real type, and* $\mathbb{H}$ *for* V *of quaternionic type, which motivates the names above.*

Hint. Show that the complexification $V_{\mathbb{C}}$ *of* V *decomposes as* $V \oplus V^*$*. Use this to compute the dimension of* $\mathrm{End}_{\mathbb{R}[G]} V$ *in all three cases. Using the fact that* $\mathbb{C} \subset \mathrm{End}_{\mathbb{R}[G]} V$*, prove the result in the complex case. In the remaining two cases, let* B *be the invariant bilinear form on* V*, and* $(,)$ *the invariant positive Hermitian form (they are defined up to a nonzero complex scalar and a positive real scalar, respectively), and define the operator* $j : V \to V$ *such that* $B(v, w) = (v, jw)$*. Show that* j *is complex antilinear (*$ji = -ij$*), and* $j^2 = \lambda \cdot Id$*, where* λ *is a real number, positive in the real case and negative in the quaternionic case (if* B *is renormalized,* j *multiplies by a nonzero complex number* z *and* j^2 *by* $z\bar{z}$*, as* j *is antilinear). Thus* j *can be normalized so that* $j^2 = 1$ *for the real case, and* $j^2 = -1$ *in the quaternionic case. Deduce the claim from this.*

(b) Show that V *is of real type if and only if* V *is the complexification of a representation* $V_{\mathbb{R}}$ *over the field of real numbers.*

Example 4.2. For $\mathbb{Z}/n\mathbb{Z}$ all irreducible representations are of complex type, except the trivial one and, if n is even, the "sign" representation, $m \to (-1)^m$, which are of real type. For S_3 all three irreducible representations $\mathbb{C}_+, \mathbb{C}_-, \mathbb{C}^2$ are of real type. For S_4 there are five irreducible representations $\mathbb{C}_+$, $\mathbb{C}_-$, $\mathbb{C}^2$, $\mathbb{C}^3_+$, $\mathbb{C}^3_-$, which are all of real type. Similarly, all five irreducible representations of A_5 – $\mathbb{C}$, $\mathbb{C}^3_+$, $\mathbb{C}^3_-$, $\mathbb{C}^4$, $\mathbb{C}^5$ are of real type. As for Q_8, its one-dimensional representations are of real type, and the two-dimensional one is of quaternionic type.

Definition 4.3. The Frobenius-Schur indicator $FS(V)$ of an irreducible representation V is 0 if it is of complex type, 1 if it is of real type, and -1 if it is of quaternionic type.

Theorem 4.4. *(Frobenius-Schur) The number of involutions (=elements of order* ≤ 2*) in* G *is equal to* $\sum_V \dim(V) FS(V)$*, i.e., the sum of dimensions of all representations of* G *of real type minus the sum of dimensions of its representations of quaternionic type.*

Proof. Let $A : V \to V$ have eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. We have

$$\mathrm{Tr}|_{S^2V}(A \otimes A) = \sum_{i \leq j} \lambda_i \lambda_j$$

$$\mathrm{Tr}|_{\Lambda^2V}(A \otimes A) = \sum_{i < j} \lambda_i \lambda_j$$

Thus,

$$\mathrm{Tr}|_{S^2V}(A \otimes A) - \mathrm{Tr}|_{\Lambda^2V}(A \otimes A) = \sum_{1 \leq i \leq n} \lambda_i^2 = \mathrm{Tr}(A^2).$$

Thus for $g \in G$ we have

$$\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\Lambda^2V}(g)$$

Therefore,

$$|G|^{-1}\chi_V(\sum_{g\in G} g^2) = \chi_{S^2V}(P) - \chi_{\wedge^2V}(P) = \dim(S^2V)^G - \dim(\wedge^2V)^G = \begin{cases} 1, & \text{if } V \text{ is of real type} \\ -1, & \text{if } V \text{ is of quaternionic type} \\ 0, & \text{if } V \text{ is of complex type} \end{cases}$$

Finally, the number of involutions in G equals

$$\frac{1}{|G|}\sum_V \dim V \chi_V(\sum_{g\in G} g^2) = \sum_{\text{real } V} \dim V - \sum_{\text{quat } V} \dim V.$$

□

Corollary 4.5. *Assume that all representations of a finite group G are defined over real numbers (i.e., all complex representations of G are obtained by complexifying real representations). Then the sum of dimensions of irreducible representations of G equals the number of involutions in G.*

Exercise. Show that any nontrivial finite group of odd order has an irreducible representation which is not defined over $\mathbb{R}$ (i.e., not realizable by real matrices).

4.2 Frobenius determinant

Enumerate the elements of a finite group G as follows: $g_1, g_2, \ldots, g_n$. Introduce n variables indexed with the elements of G :

$$x_{g_1}, x_{g_2}, \ldots, x_{g_n}.$$

Definition 4.6. Consider the matrix X_G with entries $a_{ij} = x_{g_i g_j}$. The determinant of X_G is some polynomial of degree n of $x_{g_1}, x_{g_2}, \ldots, x_{g_n}$ that is called *the Frobenius determinant.*

The following theorem, discovered by Dedekind and proved by Frobenius, became the starting point for creation of representation theory (see [Cu]).

Theorem 4.7.

$$\det X_G = \prod_{j=1}^{r} P_j(\mathbf{x})^{\deg P_j}$$

for some pairwise non-proportional irreducible polynomials $P_j(\mathbf{x})$, where r is the number of conjugacy classes of G.

We will need the following simple lemma.

Lemma 4.8. *Let Y be an $n \times n$ matrix with entries y_{ij}. Then $\det Y$ is an irreducible polynomial of $\{y_{ij}\}$.*

Proof. Let $Y = t \cdot Id + \sum_{i=1}^{n} x_i E_{i,i+1}$, where $i+1$ is computed modulo n, and $E_{i,j}$ are the elementary matrices. Then $\det(Y) = t^n - (-1)^n x_1 ... x_n$, which is obviously irreducible. Hence $\det(Y)$ is irreducible (since factors of a homogeneous polynomial are homogeneous). □

Now we are ready to proceed to the proof of Theorem 4.7.

Proof. Let $V = \mathbb{C}[G]$ be the regular representation of G. Consider the operator-valued polynomial

$$L(\mathbf{x}) = \sum_{g \in G} x_g \rho(g),$$

where $\rho(g) \in \mathrm{End}V$ is induced by g. The action of $L(\mathbf{x})$ on an element $h \in G$ is

$$L(\mathbf{x})h = \sum_{g \in G} x_g \rho(g)h = \sum_{g \in G} x_g gh = \sum_{z \in G} x_{zh^{-1}} z$$

So the matrix of the linear operator $L(\mathbf{x})$ in the basis $g_1, g_2, \ldots, g_n$ is X_G with permuted columns and hence has the same determinant up to sign.

Further, by Maschke's theorem, we have

$$\det{}_V L(\mathbf{x}) = \prod_{i=1}^{r} (\det{}_{V_i} L(\mathbf{x}))^{\dim V_i},$$

where V_i are the irreducible representations of G. We set $P_i = \det_{V_i} L(\mathbf{x})$. Let $\{e_{im}\}$ be bases of V_i and $E_{i,jk} \in \mathrm{End}\, V_i$ be the matrix units in these bases. Then $\{E_{i,jk}\}$ is a basis of $\mathbb{C}[G]$ and

$$L(\mathbf{x})|_{V_i} = \sum_{j,k} y_{i,jk} E_{i,jk},$$

where $y_{i,jk}$ are new coordinates on $\mathbb{C}[G]$ related to x_g by a linear transformation. Then

$$P_i(\mathbf{x}) = \det|_{V_i} L(\mathbf{x}) = \det(y_{i,jk})$$

Hence, P_i are irreducible (by Lemma 4.8) and not proportional to each other (as they depend on different collections of variables $y_{i,jk}$). The theorem is proved. □

4.3 Algebraic numbers and algebraic integers

We are now passing to deeper results in representation theory of finite groups. These results require the theory of algebraic numbers, which we will now briefly review.

Definition 4.9. $z \in \mathbb{C}$ is an **algebraic number** (respectively, an **algebraic integer**), if z is a root of a monic polynomial with rational (respectively, integer) coefficients.

Definition 4.10. $z \in \mathbb{C}$ is an **algebraic number**, (respectively, an **algebraic integer**), if z is an eigenvalue of a matrix with rational (respectively, integer) entries.

Proposition 4.11. *Definitions (4.9) and (4.10) are equivalent.*

Proof. To show (4.10) $\Rightarrow$ (4.9), notice that z is a root of the characteristic polynomial of the matrix (a monic polynomial with rational, respectively integer, coefficients).
To show (4.9) $\Rightarrow$ (4.10), suppose z is a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n.$$

Then the characteristic polynomial of the following matrix (called the **companion matrix**) is $p(x)$:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -a_{n-2} \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & -a_1 \end{pmatrix}.$$

Since z is a root of the characteristic polynomial of this matrix, it is an eigenvalue of this matrix. □

The set of algebraic numbers is denoted by $\overline{\mathbb{Q}}$, and the set of algebraic integers by $\mathbb{A}$.

Proposition 4.12. *(i) $\mathbb{A}$ is a ring.*

(ii) $\overline{\mathbb{Q}}$ is a field. Namely, it is an algebraic closure of the field of rational numbers.

Proof. We will be using definition (4.10). Let α be an eigenvalue of

$$\mathcal{A} \in \mathrm{Mat}_n(\mathbb{C})$$

with eigenvector v, let β be an eigenvalue of

$$\mathcal{B} \in \mathrm{Mat}_m(\mathbb{C})$$

with eigenvector w. Then $\alpha \pm \beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathrm{Id}_m \pm \mathrm{Id}_n \otimes \mathcal{B},$$

and $\alpha\beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathcal{B}.$$

The corresponding eigenvector is in both cases $v \otimes w$. This shows that both $\mathbb{A}$ and $\overline{\mathbb{Q}}$ are rings. To show that the latter is a field, it suffices to note that if $\alpha \neq 0$ is a root of a polynomial $p(x)$ of degree d, then α^{-1} is a root of $x^d p(1/x)$. The last statement is easy, since a number α is algebraic if and only if it defines a finite extension of $\mathbb{Q}$. □

Proposition 4.13. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

Proof. We will be using definition (4.9). Let z be a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n,$$

and suppose

$$z = \frac{p}{q} \in \mathbb{Q}, \gcd(p, q) = 1.$$

Notice that the leading term of $p(x)$ will have q^n in the denominator, whereas all the other terms will have a lower power of q there. Thus, if $q \neq \pm 1$, then $p(z) \notin \mathbb{Z}$, a contradiction. Thus, $z \in \mathbb{A} \cap \mathbb{Q} \Rightarrow z \in \mathbb{Z}$. The reverse inclusion follows because $n \in \mathbb{Z}$ is a root of $x - n$. □

Every algebraic number α has a **minimal polynomial** $p(x)$, which is the monic polynomial with rational coefficients of the smallest degree such that $p(\alpha) = 0$. Any other polynomial $q(x)$ with rational coefficients such that $q(\alpha) = 0$ is divisible by $p(x)$. Roots of $p(x)$ are called the **algebraic conjugates** of α; they are roots of any polynomial q with rational coefficients such that $q(\alpha) = 0$.

Note that any algebraic conjugate of an algebraic integer is obviously also an algebraic integer. Therefore, by the Vieta theorem, the minimal polynomial of an algebraic integer has integer coefficients.

Below we will need the following lemma:

Lemma 4.14. *If $\alpha_1, ..., \alpha_m$ are algebraic numbers, then all algebraic conjugates to $\alpha_1 + ... + \alpha_m$ are of the form $\alpha_1' + ... + \alpha_m'$, where α_i' are some algebraic conjugates of α_i.*

Proof. It suffices to prove this for two summands. If α_i are eigenvalues of rational matrices A_i of smallest size (i.e., their characteristic polynomials are the minimal polynomials of α_i), then $\alpha_1+\alpha_2$ is an eigenvalue of $A := A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$. Therefore, so is any algebraic conjugate to $\alpha_1 + \alpha_2$. But all eigenvalues of A are of the form $\alpha_1' + \alpha_2'$, so we are done. □

Problem 4.15. *(a) Show that for any finite group G there exists a finite Galois extension $K \subset \mathbb{C}$ of $\mathbb{Q}$ such that any finite dimensional complex representation of G has a basis in which the matrices of the group elements have entries in K.*

Hint. Consider the representations of G over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

(b) Show that if V is an irreducible complex representation of a finite group G of dimension > 1 then there exists $g \in G$ such that $\chi_V(g) = 0$.

Hint: Assume the contrary. Use orthonormality of characters to show that the arithmetic mean of the numbers $|\chi_V(g)|^2$ for $g \neq 1$ is < 1. Deduce that their product β satisfies $0 < \beta < 1$. Show that all conjugates of β satisfy the same inequalities (consider the Galois conjugates of the representation V, i.e. representations obtained from V by the action of the Galois group of K over $\mathbb{Q}$ on the matrices of group elements in the basis from part (a)). Then derive a contradiction.

Remark. *Here is a modification of this argument, which does not use (a). Let $N = |G|$. For any $0 < j < N$ coprime to N, show that the map $g \mapsto g^j$ is a bijection $G \to G$. Deduce that $\prod_{g=1} |\chi_V(g^j)|^2 = \beta$. Then show that $\beta \in K := \mathbb{Q}(\zeta)$, $\zeta = e^{2\pi i/N}$, and does not change under the automorphism of K given by $\zeta \mapsto \zeta^j$. Deduce that β is an integer, and derive a contradiction.*

4.4 Frobenius divisibility

Theorem 4.16. *Let G be a finite group, and let V be an irreducible representation of G over $\mathbb{C}$. Then*

$$\dim V \text{ divides } |G|.$$

Proof. Let $C_1, C_2, \dots, C_n$ be the conjugacy classes of G. Set

$$\lambda_i = \chi_V(g_{C_i}) \frac{|C_i|}{\dim V},$$

where g_{C_i} is a representative of C_i.

Proposition 4.17. *The numbers λ_i are algebraic integers for all i.*

Proof. Let C be a conjugacy class in G, and $P = \sum_{h \in C} h$. Then P is a central element of $\mathbb{Z}[G]$, so it acts on V by some scalar λ, which is an algebraic integer (indeed, since $\mathbb{Z}[G]$ is a finitely generated $\mathbb{Z}$-module, any element of $\mathbb{Z}[G]$ is integral over $\mathbb{Z}$, i.e., satisfies a monic polynomial equation with integer coefficients). On the other hand, taking the trace of P in V, we get $|C|\chi_V(g) = \lambda \dim V$, $g \in C$, so $\lambda = \frac{|C|\chi_V(g)}{\dim V}$. □

Now, consider

$$\sum_i \lambda_i \overline{\chi_V(g_{C_i})}.$$

This is an algebraic integer, since:

(i) λ_i are algebraic integers by Proposition 4.17,

(ii) $\chi_V(g_{C_i})$ is a sum of roots of unity (it is the sum of eigenvalues of the matrix of $\rho(g_{C_i})$, and since $g_{C_i}^{|G|} = e$ in G, the eigenvalues of $\rho(g_{C_i})$ are roots of unity), and

(iii) $\mathbb{A}$ is a ring (Proposition 4.12).

On the other hand, from the definition of λ_i,

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \sum_i \frac{|C_i|\chi_V(g_{C_i})\overline{\chi_V(g_{C_i})}}{\dim V}.$$

Recalling that χ_V is a class function, this is equal to

$$\sum_{g\in G} \frac{\chi_V(g)\overline{\chi_V(g)}}{\dim V} = \frac{|G|(\chi_V, \chi_V)}{\dim V}.$$

Since V is an irreducible representation, $(\chi_V, \chi_V) = 1$, so

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \frac{|G|}{\dim V}.$$

Since $\frac{|G|}{\dim V} \in \mathbb{Q}$ and $\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} \in \mathbb{A}$, by Proposition 4.13 $\frac{|G|}{\dim V} \in \mathbb{Z}$. $\square$

4.5 Burnside's Theorem

Definition 4.18. A group G is called *solvable* if there exists a series of nested normal subgroups

$$\{e\} = G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$$

where G_{i+1}/G_i is abelian for all $1 \le i \le n-1$.

Remark 4.19. Such groups are called solvable because they first arose as Galois groups of polynomial equations which are solvable in radicals.

Theorem 4.20 (Burnside). *Any group G of order $p^a q^b$, where p and q are prime and $a, b \ge 0$, is solvable.*

This famous result in group theory was proved by the British mathematician William Burnside in the early 20-th century, using representation theory (see [Cu]). Here is this proof, presented in modern language.

Before proving Burnside's theorem we will prove several other results which are of independent interest.

Theorem 4.21. *Let V be an irreducible representation of a finite group G and let C be a conjugacy class of G with $\gcd(|C|, \dim(V)) = 1$. Then for any $g \in C$, either $\chi_V(g) = 0$ or g acts as a scalar on V.*

The proof will be based on the following lemma.

Lemma 4.22. *If $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n$ are roots of unity such that $\dfrac{1}{n}(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n)$ is an algebraic integer, then either $\varepsilon_1 = \ldots = \varepsilon_n$ or $\varepsilon_1 + \ldots + \varepsilon_n = 0$.*

Proof. Let $a = \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n)$. If not all ε_i are equal, then $|a| < 1$. Moreover, since any algebraic conjugate of a root of unity is also a root of unity, $|a'| \leq 1$ for any algebraic conjugate a' of a. But the product of all algebraic conjugates of a is an integer. Since it has absolute value < 1, it must equal zero. Therefore, $a = 0$. $\square$

Proof of theorem 4.21.

Let $\dim V = n$. Let $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ be the eigenvalues of $\rho_V(g)$. They are roots of unity, so $\chi_V(g)$ is an algebraic integer. Also, by Proposition 4.17, $\frac{1}{n}|C|\chi_V(g)$ is an algebraic integer. Since $\gcd(n, |C|) = 1$, there exist integers a, b such that $a|C| + bn = 1$. This implies that

$$\frac{\chi_V(g)}{n} = \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n).$$

is an algebraic integer. Thus, by Lemma 4.22, we get that either $\varepsilon_1 = \ldots = \varepsilon_n$ or $\varepsilon_1 + \ldots + \varepsilon_n = \chi_V(g) = 0$. In the first case, since $\rho_V(g)$ is diagonalizable, it must be scalar. In the second case, $\chi_V(g) = 0$. The theorem is proved.

Theorem 4.23. *Let G be a finite group, and let C be a conjugacy class in G of order p^k where p is prime and $k > 0$. Then G has a proper nontrivial normal subgroup (i.e., G is not simple).*

Proof. Choose an element $g \in C$. Since $g \neq e$, by orthogonality of columns of the character table,

$$\sum_{V \in \mathrm{Irr}G} \dim V \chi_V(g) = 0. \tag{4}$$

We can divide $\mathrm{Irr}G$ into three parts:

1. the trivial representation,
2. D, the set of irreducible representations whose dimension is divisible by p, and
3. N, the set of non-trivial irreducible representations whose dimension is not divisible by p.

Lemma 4.24. *There exists $V \in N$ such that $\chi_V(g) = 0$.*

Proof. If $V \in D$, the number $\frac{1}{p}\dim(V)\chi_V(g)$ is an algebraic integer, so

$$a = \sum_{V \in D} \frac{1}{p} \dim(V)\chi_V(g)$$

is an algebraic integer.

Now, by (4), we have

$$0 = \chi_{\mathbb{C}}(g) + \sum_{V \in D} \dim V \chi_V(g) + \sum_{V \in N} \dim V \chi_V(g) = 1 + pa + \sum_{V \in N} \dim V \chi_V(g).$$

This means that the last summand is nonzero. $\square$

Now pick $V \in N$ such that $\chi_V(g) = 0$; it exists by Lemma 4.24. Theorem 4.21 implies that g (and hence any element of C) acts by a scalar in V. Now let H be the subgroup of G generated by elements ab^{-1}, $a, b \in C$. It is normal and acts trivially in V, so $H \neq G$, as V is nontrivial. Also $H \neq 1$, since $|C| > 1$. □

Proof of Burnside's theorem.

Assume Burnside's theorem is false. Then there exists a nonsolvable group G of order $p^a q^b$. Let G be the smallest such group. Then G is simple, and by Theorem 4.23, it cannot have a conjugacy class of order p^k or q^k, $k \geq 1$. So the order of any conjugacy class in G is either 1 or is divisible by pq. Adding the orders of conjugacy classes and equating the sum to $p^a q^b$, we see that there has to be more than one conjugacy class consisting just of one element. So G has a nontrivial center, which gives a contradiction.

4.6 Representations of products

Theorem 4.25. *Let G, H be finite groups, $\{V_i\}$ be the irreducible representations of G over a field k (of any characteristic), and $\{W_j\}$ be the irreducible representations of H over k. Then the irreducible representations of $G \times H$ over k are $\{V_i \otimes W_j\}$.*

Proof. This follows from Theorem 2.26. □

4.7 Virtual representations

Definition 4.26. A *virtual representation* of a finite group G is an integer linear combination of irreducible representations of G, $V = \sum n_i V_i$, $n_i \in \mathbb{Z}$ (i.e., n_i are not assumed to be nonnegative). The character of V is $\chi_V := \sum n_i \chi_{V_i}$.

The following lemma is often very useful (and will be used several times below).

Lemma 4.27. *Let V be a virtual representation with character χ_V. If $(\chi_V, \chi_V) = 1$ and $\chi_V(1) > 0$ then χ_V is a character of an irreducible representation of G.*

Proof. Let $V_1, V_2, \ldots, V_m$ be the irreducible representations of G, and $V = \sum n_i V_i$. Then by orthonormality of characters, $(\chi_V, \chi_V) = \sum_i n_i^2$. So $\sum_i n_i^2 = 1$, meaning that $n_i = \pm 1$ for exactly one i, and $n_j = 0$ for $j \neq i$. But $\chi_V(1) > 0$, so $n_i = +1$ and we are done. □

4.8 Induced Representations

Given a representation V of a group G and a subgroup $H \subset G$, there is a natural way to construct a representation of H. The restricted representation of V to H, $\mathrm{Res}_H^G V$ is the representation given by the vector space V and the action $\rho_{\mathrm{Res}_H^G V} = \rho_V|_H$.

There is also a natural, but more complicated way to construct a representation of a group G given a representation V of its subgroup H.

Definition 4.28. If G is a group, $H \subset G$, and V is a representation of H, then the *induced representation* $Ind_H^G V$ is the representation of G with

$$\mathrm{Ind}_H^G V = \{f : G \to V | f(hx) = \rho_V(h) f(x) \forall x \in G, h \in H\}$$

and the action $g(f)(x) = f(xg)\ \forall g \in G$.

Remark 4.29. In fact, $\mathrm{Ind}_H^G V$ is naturally isomorphic to $\mathrm{Hom}_H(k[G], V)$.

Let us check that $\mathrm{Ind}_H^G V$ is indeed a representation:

$g(f)(hx) = f(hxg) = \rho_V(h)f(xg) = \rho_V(h)g(f)(x)$, and $g(g'(f))(x) = g'(f)(xg) = f(xgg') = (gg')(f)(x)$ for any $g, g', x \in G$ and $h \in H$.

Remark 4.30. Notice that if we choose a representative x_σ from every right H-coset σ of G, then any $f \in \mathrm{Ind}_H^G V$ is uniquely determined by $\{f(x_\sigma)\}$.

Because of this,

$$\dim(\mathrm{Ind}_H^G V) = \dim V \cdot \frac{|G|}{|H|}.$$

Problem 4.31. *Check that if $K \subset H \subset G$ are groups and V a representation of K then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H V$ is isomorphic to $\mathrm{Ind}_K^G V$.*

Exercise. Let $K \subset G$ be finite groups, and $\chi : K \to \mathbb{C}^*$ be a homomorphism. Let $\mathbb{C}_\chi$ be the corresponding 1-dimensional representation of K. Let

$$e_\chi = \frac{1}{|K|} \sum_{g \in K} \chi(g)^{-1} g \in \mathbb{C}[K]$$

be the idempotent corresponding to χ. Show that the G-representation $\mathrm{Ind}_K^G \mathbb{C}_\chi$ is naturally isomorphic to $\mathbb{C}[G]e_\chi$ (with G acting by left multiplication).

4.9 The Mackey formula

Let us now compute the character χ of $\mathrm{Ind}_H^G V$. In each right coset $\sigma \in H\backslash G$, choose a representative x_σ.

Theorem 4.32. *(The Mackey formula) One has*

$$\chi(g) = \sum_{\sigma \in H\backslash G : x_\sigma g x_\sigma^{-1} \in H} \chi_V(x_\sigma g x_\sigma^{-1}).$$

Remark. If the characteristic of the ground field k is relatively prime to $|H|$, then this formula can be written as

$$\chi(g) = \frac{1}{|H|} \sum_{x \in G : xgx^{-1} \in H} \chi_V(xgx^{-1}).$$

Proof. For a right H-coset σ of G, let us define

$$V_\sigma = \{f \in \mathrm{Ind}_H^G V \mid f(g) = 0\ \forall g \notin \sigma\}.$$

Then one has

$$\mathrm{Ind}_H^G V = \bigoplus_\sigma V_\sigma,$$

and so

$$\chi(g) = \sum_\sigma \chi_\sigma(g),$$

where $\chi_\sigma(g)$ is the trace of the diagonal block of $\rho(g)$ corresponding to V_σ.

Since $g(\sigma) = \sigma g$ is a right H-coset for any right H-coset σ, $\chi_\sigma(g) = 0$ if $\sigma \neq \sigma g$.

Now assume that $\sigma = \sigma g$. Then $x_\sigma g = h x_\sigma$ where $h = x_\sigma g x_\sigma^{-1} \in H$. Consider the vector space homomorphism $\alpha : V_\sigma \to V$ with $\alpha(f) = f(x_\sigma)$. Since $f \in V_\sigma$ is uniquely determined by $f(x_\sigma)$, α is an isomorphism. We have

$$\alpha(gf) = g(f)(x_\sigma) = f(x_\sigma g) = f(h x_\sigma) = \rho_V(h) f(x_\sigma) = h\alpha(f),$$

and $gf = \alpha^{-1} h \alpha(f)$. This means that $\chi_\sigma(g) = \chi_V(h)$. Therefore

$$\chi(g) = \sum_{\sigma \in H\backslash G, \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1}).$$

$\square$

4.10 Frobenius reciprocity

A very important result about induced representations is the Frobenius Reciprocity Theorem which connects the operations Ind and Res.

Theorem 4.33. (Frobenius Reciprocity)

Let $H \subset G$ be groups, V be a representation of G and W a representation of H. Then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ is naturally isomorphic to $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

Proof. Let $E = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ and $E' = \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$. Define $F : E \to E'$ and $F' : E' \to E$ as follows: $F(\alpha)v = (\alpha v)(e)$ for any $\alpha \in E$ and $(F'(\beta)v)(x) = \beta(xv)$ for any $\beta \in E'$.

In order to check that F and F' are well defined and inverse to each other, we need to check the following five statements.

Let $\alpha \in E$, $\beta \in E'$, $v \in V$, and $x, g \in G$.

(a) $F(\alpha)$ is an H-homomorphism, i.e., $F(\alpha)hv = hF(\alpha)v$.

Indeed, $F(\alpha)hv = (\alpha hv)(e) = (h\alpha v)(e) = (\alpha v)(he) = (\alpha v)(eh) = h \cdot (\alpha v)(e) = hF(\alpha)v$.

(b) $F'(\beta)v \in \mathrm{Ind}_H^G W$, i.e., $(F'(\beta)v)(hx) = h(F'(\beta)v)(x)$.

Indeed, $(F'(\beta)v)(hx) = \beta(hxv) = h\beta(xv) = h(F'(\beta)v)(x)$.

(c) $F'(\beta)$ is a G-homomorphism, i.e. $F'(\beta)gv = g(F'(\beta)v)$.

Indeed, $(F'(\beta)gv)(x) = \beta(xgv) = (F'(\beta)v)(xg) = (g(F'(\beta)v))(x)$.

(d) $F \circ F' = Id_{E'}$.

This holds since $F(F'(\beta))v = (F'(\beta)v)(e) = \beta(v)$.

(e) $F' \circ F = Id_E$, i.e., $(F'(F(\alpha))v)(x) = (\alpha v)(x)$.

Indeed, $(F'(F(\alpha))v)(x) = F(\alpha xv) = (\alpha xv)(e) = (x\alpha v)(e) = (\alpha v)(x)$, and we are done. $\square$

Exercise. The purpose of this exercise is to understand the notions of restricted and induced representations as part of a more advanced framework. This framework is the notion of tensor products over k-algebras (which generalizes the tensor product over k which we defined in Definition

1.48). In particular, this understanding will lead us to a new proof of the Frobenius reciprocity and to some analogies between induction and restriction.

Throughout this exercise, we will use the notation and results of the Exercise in Section 1.10.

Let G be a finite group and $H \subset G$ a subgroup. We consider $k[G]$ as a $(k[H], k[G])$-bimodule (both module structures are given by multiplication inside $k[G]$). We denote this bimodule by $k[G]_1$. On the other hand, we can also consider $k[G]$ as a $(k[G], k[H])$-bimodule (again, both module structures are given by multiplication). We denote this bimodule by $k[G]_2$.

(a) Let V be a representation of G. Then, V is a left $k[G]$-module, thus a $(k[G], k)$-bimodule. Thus, the tensor product $k[G]_1 \otimes_{k[G]} V$ is a $(k[H], k)$-bimodule, i. e., a left $k[H]$-module. Prove that this tensor product is isomorphic to $\mathrm{Res}_H^G V$ as a left $k[H]$-module. The isomorphism $\mathrm{Res}_H^G V \to k[G]_1 \otimes_{k[G]} V$ is given by $v \mapsto 1 \otimes_{k[G]} v$ for every $v \in \mathrm{Res}_H^G V$.

(b) Let W be a representation of H. Then, W is a left $k[H]$-module, thus a $(k[H], k)$-bimodule. Then, $\mathrm{Ind}_H^G W \cong \mathrm{Hom}_H(k[G], W)$, according to Remark 4.30. In other words, $\mathrm{Ind}_H^G W \cong \mathrm{Hom}_{k[H]}(k[G]_1, W)$. Now, use part (b) of the Exercise in Section 1.10 to conclude Theorem 4.33.

(c) Let V be a representation of G. Then, V is a left $k[G]$-module, thus a $(k[G], k)$-bimodule. Prove that not only $k[G]_1 \otimes_{k[G]} V$, but also $\mathrm{Hom}_{k[G]}(k[G]_2, V)$ is isomorphic to $\mathrm{Res}_H^G V$ as a left $k[H]$-module. The isomorphism $\mathrm{Hom}_{k[G]}(k[G]_2, V) \to \mathrm{Res}_H^G V$ is given by $f \mapsto f(1)$ for every $f \in \mathrm{Hom}_{k[G]}(k[G]_2, V)$.

(d) Let W be a representation of H. Then, W is a left $k[H]$-module, thus a $(k[H], k)$-bimodule. Show that $\mathrm{Ind}_H^G W$ is not only isomorphic to $\mathrm{Hom}_{k[H]}(k[G]_1, W)$, but also isomorphic to $k[G]_2 \otimes_{k[H]} W$. The isomorphism $\mathrm{Hom}_{k[H]}(k[G]_1, W) \to k[G]_2 \otimes_{k[H]} W$ is given by $f \mapsto \sum_{g \in P} g^{-1} \otimes_{k[H]} f(g)$ for every $f \in \mathrm{Hom}_{k[H]}(k[G]_1, W)$, where P is a set of distinct representatives for the right H-cosets in G. (This isomorphism is independent of the choice of representatives.)

(e) Let V be a representation of G and W a representation of H. Use (b) to prove that $\mathrm{Hom}_G\left(\mathrm{Ind}_H^G W, V\right)$ is naturally isomorphic to $\mathrm{Hom}_H\left(W, \mathrm{Res}_H^G V\right)$.

(f) Let V be a representation of H. Prove that $\mathrm{Ind}_H^G(V^*) \cong \left(\mathrm{Ind}_H^G V\right)^*$ as representations of G. [Hint: Write $\mathrm{Ind}_H^G V$ as $k[G]_2 \otimes_{k[H]} V$ and write $\mathrm{Ind}_H^G(V^*)$ as $\mathrm{Hom}_{k[H]}(k[G]_1, V^*)$. Prove that the map $\mathrm{Hom}_{k[H]}(k[G]_1, V^*) \times \left(\mathrm{Ind}_H^G(V^*)\right) \to k$ given by $\left(f, \left(x \otimes_{k[H]} v\right)\right) \mapsto (f(Sx))(v)$ is a nondegenerate G-invariant bilinear form, where $S : k[G] \to k[G]$ is the linear map defined by $Sg = g^{-1}$ for every $g \in G$.]

4.11 Examples

Here are some examples of induced representations (we use the notation for representations from the character tables).

1. Let $G = S_3$, $H = \mathbb{Z}_2$. Using the Frobenius reciprocity, we obtain: $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}^2 \oplus \mathbb{C}_+$, $\mathrm{Ind}_H^G \mathbb{C}_- = \mathbb{C}^2 \oplus \mathbb{C}_-$.

2. Let $G = S_3$, $H = \mathbb{Z}_3$. Then we obtain $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}_+ \oplus \mathbb{C}_-$, $\mathrm{Ind}_H^G \mathbb{C}_\epsilon = \mathrm{Ind}_H^G \mathbb{C}_{\epsilon^2} = \mathbb{C}^2$.

3. Let $G = S_4$, $H = S_3$. Then $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}_+ \oplus \mathbb{C}_-^3$, $\mathrm{Ind}_H^G \mathbb{C}_- = \mathbb{C}_- \oplus \mathbb{C}_+^3$, $\mathrm{Ind}_H^G \mathbb{C}^2 = \mathbb{C}^2 \oplus \mathbb{C}_-^3 \oplus \mathbb{C}_+^3$.

Problem 4.34. *Compute the decomposition into irreducibles of all the representations of A_5 induced from all the irreducible representations of*

(a) $\mathbb{Z}_2$

(b) $\mathbb{Z}_3$

(c) $\mathbb{Z}_5$

(d) A_4

(e) $\mathbb{Z}_2 \times \mathbb{Z}_2$

4.12 Representations of S_n

In this subsection we give a description of the representations of the symmetric group S_n for any n.

Definition 4.35. A **partition** λ of n is a representation of n in the form $n = \lambda_1 + \lambda_2 + ... + \lambda_p$, where λ_i are positive integers, and $\lambda_i \geq \lambda_{i+1}$.

To such λ we will attach a **Young diagram** Y_λ, which is the union of rectangles $-i \leq y \leq -i+1$, $0 \leq x \leq \lambda_i$ in the coordinate plane, for $i = 1, ..., p$. Clearly, Y_λ is a collection of n unit squares. A **Young tableau** corresponding to Y_λ is the result of filling the numbers $1, ..., n$ into the squares of Y_λ in some way (without repetitions). For example, we will consider the Young tableau T_λ obtained by filling in the numbers in the increasing order, left to right, top to bottom.

We can define two subgroups of S_n corresponding to T_λ:

1. The row subgroup P_λ: the subgroup which maps every element of $\{1, ..., n\}$ into an element standing in the same row in T_λ.

2. The column subgroup Q_λ: the subgroup which maps every element of $\{1, ..., n\}$ into an element standing in the same column in T_λ.

Clearly, $P_\lambda \cap Q_\lambda = \{1\}$.

Define the *Young projectors:*

$$a_\lambda := \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g,$$

$$b_\lambda := \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} (-1)^g g,$$

where $(-1)^g$ denotes the sign of the permutation g. Set $c_\lambda = a_\lambda b_\lambda$. Since $P_\lambda \cap Q_\lambda = \{1\}$, this element is nonzero.

The irreducible representations of S_n are described by the following theorem.

Theorem 4.36. *The subspace $V_\lambda := \mathbb{C}[S_n]c_\lambda$ of $\mathbb{C}[S_n]$ is an irreducible representation of S_n under left multiplication. Every irreducible representation of S_n is isomorphic to V_λ for a unique λ.*

The modules V_λ are called the **Specht modules**.

The proof of this theorem is given in the next subsection.

Example 4.37.

For the partition $\lambda = (n)$, $P_\lambda = S_n$, $Q_\lambda = \{1\}$, so c_λ is the symmetrizer, and hence V_λ is the trivial representation.

For the partition $\lambda = (1, ..., 1)$, $Q_\lambda = S_n$, $P_\lambda = \{1\}$, so c_λ is the antisymmetrizer, and hence V_λ is the sign representation.

$n = 3$. For $\lambda = (2, 1)$, $V_\lambda = \mathbb{C}^2$.

$n = 4$. For $\lambda = (2, 2)$, $V_\lambda = \mathbb{C}^2$; for $\lambda = (3, 1)$, $V_\lambda = \mathbb{C}^3_-$; for $\lambda = (2, 1, 1)$, $V_\lambda = \mathbb{C}^3_+$.

Corollary 4.38. *All irreducible representations of S_n can be given by matrices with rational entries.*

Problem 4.39. *Find the sum of dimensions of all irreducible representations of the symmetric group S_n.*

Hint. Show that all irreducible representations of S_n are real, i.e., admit a nondegenerate invariant symmetric form. Then use the Frobenius-Schur theorem.

4.13 Proof of Theorem 4.36

Lemma 4.40. *Let $x \in \mathbb{C}[S_n]$. Then $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where ℓ_λ is a linear function.*

Proof. If $g \in P_\lambda Q_\lambda$, then g has a unique representation as pq, $p \in P_\lambda, q \in Q_\lambda$, so $a_\lambda g b_\lambda = (-1)^q c_\lambda$. Thus, to prove the required statement, we need to show that if g is a permutation which is not in $P_\lambda Q_\lambda$ then $a_\lambda g b_\lambda = 0$.

To show this, it is sufficient to find a transposition t such that $t \in P_\lambda$ and $g^{-1} t g \in Q_\lambda$; then

$$a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g (g^{-1} t g) b_\lambda = -a_\lambda g b_\lambda,$$

so $a_\lambda g b_\lambda = 0$. In other words, we have to find two elements i, j standing in the same row in the tableau $T = T_\lambda$, and in the same column in the tableau $T' = gT$ (where gT is the tableau of the same shape as T obtained by permuting the entries of T by the permutation g). Thus, it suffices to show that if such a pair does not exist, then $g \in P_\lambda Q_\lambda$, i.e., there exists $p \in P_\lambda$, $q' \in Q'_\lambda := g Q_\lambda g^{-1}$ such that $pT = q'T'$ (so that $g = pq^{-1}, q = g^{-1} q' g \in Q_\lambda$).

Any two elements in the first row of T must be in different columns of T', so there exists $q'_1 \in Q'_\lambda$ which moves all these elements to the first row. So there is $p_1 \in P_\lambda$ such that $p_1 T$ and $q'_1 T'$ have the same first row. Now do the same procedure with the second row, finding elements p_2, q'_2 such that $p_2 p_1 T$ and $q'_2 q'_1 T'$ have the same first two rows. Continuing so, we will construct the desired elements p, q'. The lemma is proved. ∎

Let us introduce the **lexicographic ordering** on partitions: $\lambda > \mu$ if the first nonvanishing $\lambda_i - \mu_i$ is positive.

Lemma 4.41. *If $\lambda > \mu$ then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.*

Proof. Similarly to the previous lemma, it suffices to show that for any $g \in S_n$ there exists a transposition $t \in P_\lambda$ such that $g^{-1} t g \in Q_\mu$. Let $T = T_\lambda$ and $T' = gT_\mu$. We claim that there are two integers which are in the same row of T and the same column of T'. Indeed, if $\lambda_1 > \mu_1$, this is clear by the pigeonhole principle (already for the first row). Otherwise, if $\lambda_1 = \mu_1$, like in the proof of the previous lemma, we can find elements $p_1 \in P_\lambda, q'_1 \in g Q_\mu g^{-1}$ such that $p_1 T$ and $q'_1 T'$ have the same first row, and repeat the argument for the second row, and so on. Eventually, having done $i - 1$ such steps, we'll have $\lambda_i > \mu_i$, which means that some two elements of the i-th row of the first tableau are in the same column of the second tableau, completing the proof. ∎

Lemma 4.42. *c_λ is proportional to an idempotent. Namely, $c_\lambda^2 = \frac{n!}{\dim V_\lambda} c_\lambda$.*

Proof. Lemma 4.40 implies that c_λ^2 is proportional to c_λ. Also, it is easy to see that the trace of c_λ in the regular representation is $n!$ (as the coefficient of the identity element in c_λ is 1). This implies the statement. $\square$

Lemma 4.43. *Let A be an algebra and e be an idempotent in A. Then for any left A-module M, one has $Hom_A(Ae, M) \cong eM$ (namely, $x \in eM$ corresponds to $f_x : Ae \to M$ given by $f_x(a) = ax$, $a \in Ae$).*

Proof. Note that $1 - e$ is also an idempotent in A. Thus the statement immediately follows from the fact that $\mathrm{Hom}_A(A, M) \cong M$ and the decomposition $A = Ae \oplus A(1 - e)$. $\square$

Now we are ready to prove Theorem 4.36. Let $\lambda \geq \mu$. Then by Lemmas 4.42, 4.43

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \mathrm{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu) = c_\lambda \mathbb{C}[S_n]c_\mu.$$

The latter space is zero for $\lambda > \mu$ by Lemma 4.41, and 1-dimensional if $\lambda = \mu$ by Lemmas 4.40 and 4.42. Therefore, V_λ are irreducible, and V_λ is not isomorphic to V_μ if $\lambda = \mu$. Since the number of partitions equals the number of conjugacy classes in S_n, the representations V_λ exhaust all the irreducible representations of S_n. The theorem is proved.

4.14 Induced representations for S_n

Denote by U_λ the representation $\mathrm{Ind}_{P_\lambda}^{S_n}\mathbb{C}$. It is easy to see that U_λ can be alternatively defined as $U_\lambda = \mathbb{C}[S_n]a_\lambda$.

Proposition 4.44. *$Hom(U_\lambda, V_\mu) = 0$ for $\mu < \lambda$, and $\dim Hom(U_\lambda, V_\lambda) = 1$. Thus, $U_\lambda = \oplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu$, where $K_{\mu\lambda}$ are nonnegative integers and $K_{\lambda\lambda} = 1$.*

Definition 4.45. The integers $K_{\mu\lambda}$ are called the **Kostka numbers**.

Proof. By Lemmas 4.42 and 4.43,

$$\mathrm{Hom}(U_\lambda, V_\mu) = \mathrm{Hom}(\mathbb{C}[S_n]a_\lambda, \mathbb{C}[S_n]a_\mu b_\mu) = a_\lambda \mathbb{C}[S_n]a_\mu b_\mu,$$

and the result follows from Lemmas 4.40 and 4.41. $\square$

Now let us compute the character of U_λ. Let $C_\mathbf{i}$ be the conjugacy class in S_n having i_l cycles of length l for all $l \geq 1$ (here $\mathbf{i}$ is a shorthand notation for $(i_1, ..., i_l, ...)$). Also let $x_1, ..., x_N$ be variables, and let

$$H_m(x) = \sum_i x_i^m$$

be the power sum polynomials.

Theorem 4.46. *Let $N \geq p$ (where p is the number of parts of λ). Then $\chi_{U_\lambda}(C_\mathbf{i})$ is the coefficient[6] of $x^\lambda := \prod x_j^{\lambda_j}$ in the polynomial*

$$\prod_{m \geq 1} H_m(x)^{i_m}.$$

[6] If $j > p$, we define λ_j to be zero.

Proof. The proof is obtained easily from the Mackey formula. Namely, $\chi_{U_\lambda}(C_{\mathbf{i}})$ is the number of elements $x \in S_n$ such that $xgx^{-1} \in P_\lambda$ (for a representative $g \in C_{\mathbf{i}}$), divided by $|P_\lambda|$. The order of P_λ is $\prod_i \lambda_i!$, and the number of elements x such that $xgx^{-1} \in P_\lambda$ is the number of elements in P_λ conjugate to g (i.e. $|C_{\mathbf{i}} \cap P_\lambda|$) times the order of the centralizer Z_g of g (which is $n!/|C_{\mathbf{i}}|$). Thus,

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \frac{|Z_g|}{\prod_j \lambda_j!}|C_{\mathbf{i}} \cap P_\lambda|.$$

Now, it is easy to see that the centralizer Z_g of g is isomorphic to $\prod_m S_{i_m} \ltimes (\mathbb{Z}/m\mathbb{Z})^{i_m}$, so

$$|Z_g| = \prod_m m^{i_m} i_m!,$$

and we get

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \frac{\prod_m m^{i_m} i_m!}{\prod_j \lambda_j!}|C_{\mathbf{i}} \cap P_\lambda|.$$

Now, since $P_\lambda = \prod_j S_{\lambda_j}$, we have

$$|C_{\mathbf{i}} \cap P_\lambda| = \sum_r \prod_{j \geq 1} \frac{\lambda_j!}{\prod_{m \geq 1} m^{r_{jm}} r_{jm}!},$$

where $r = (r_{jm})$ runs over all collections of nonnegative integers such that

$$\sum_m m r_{jm} = \lambda_j, \sum_j r_{jm} = i_m.$$

Indeed, an element of $C_{\mathbf{i}}$ that is in P_λ would define an ordered partition of each λ_j into parts (namely, cycle lengths), with m occuring r_{jm} times, such that the total (over all j) number of times each part m occurs is i_m. Thus we get

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \sum_r \prod_m \frac{i_m!}{\prod_j r_{jm}!}$$

But this is exactly the coefficient of x^λ in

$$\prod_{m \geq 1} (x_1^m + ... + x_N^m)^{i_m}$$

(r_{jm} is the number of times we take x_j^m). $\square$

4.15 The Frobenius character formula

Let $\Delta(x) = \prod_{1 \leq i < j \leq N}(x_i - x_j)$. Let $\rho = (N-1, N-2, ..., 0) \in \mathbb{C}^N$. The following theorem, due to Frobenius, gives a character formula for the Specht modules V_λ.

Theorem 4.47. *Let $N \geq p$. Then $\chi_{V_\lambda}(C_{\mathbf{i}})$ is the coefficient of $x^{\lambda+\rho} := \prod x_j^{\lambda_j+N-j}$ in the polynomial*

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m}.$$

Remark. Here is an equivalent formulation of Theorem 4.47: $\chi_{V_\lambda}(C_{\mathbf{i}})$ is the coefficient of x^λ in the (Laurent) polynomial

$$\prod_{i<j}\left(1 - \frac{x_j}{x_i}\right) \prod_{m \geq 1} H_m(x)^{i_m}.$$

Proof. Denote χ_{V_λ} shortly by χ_λ. Let us denote the class function defined in the theorem by θ_λ. We claim that this function has the property $\theta_\lambda = \sum_{\mu \geq \lambda} L_{\mu\lambda}\chi_\mu$, where $L_{\mu\lambda}$ are integers and $L_{\lambda\lambda} = 1$. Indeed, from Theorem 4.46 we have

$$\theta_\lambda = \sum_{\sigma \in S_N} (-1)^\sigma \chi_{U_{\lambda+\rho-\sigma(\rho)}},$$

where if the vector $\lambda + \rho - \sigma(\rho)$ has a negative entry, the corresponding term is dropped, and if it has nonnegative entries which fail to be nonincreasing, then the entries should be reordered in the nonincreasing order, making a partition that we'll denote $\langle \lambda + \rho - \sigma(\rho) \rangle$ (i.e., we agree that $U_{\lambda+\rho-\sigma(\rho)} := U_{\langle \lambda+\rho-\sigma(\rho)\rangle}$). Now note that $\mu = \langle \lambda + \rho - \sigma(\rho) \rangle$ is obtained from λ by adding vectors of the form $e_i - e_j$, $i < j$, which implies that $\mu > \lambda$ or $\mu = \lambda$, and the case $\mu = \lambda$ arises only if $\sigma = 1$, as desired.

Therefore, to show that $\theta_\lambda = \chi_\lambda$, by Lemma 4.27, it suffices to show that $(\theta_\lambda, \theta_\lambda) = 1$.

We have

$$(\theta_\lambda, \theta_\lambda) = \frac{1}{n!} \sum_{\mathbf{i}} |C_{\mathbf{i}}| \theta_\lambda(C_{\mathbf{i}})^2.$$

Using that

$$|C_{\mathbf{i}}| = \frac{n!}{\prod_m m^{i_m} i_m!},$$

we conclude that $(\theta_\lambda, \theta_\lambda)$ is the coefficient of $x^{\lambda+\rho} y^{\lambda+\rho}$ in the series $R(x, y) = \Delta(x)\Delta(y)S(x, y)$, where

$$S(x, y) = \sum_{\mathbf{i}} \prod_m \frac{(\sum_j x_j^m)^{i_m} (\sum_k y_k^m)^{i_m}}{m^{i_m} i_m!} = \sum_{\mathbf{i}} \prod_m \frac{(\sum_{j,k} x_j^m y_k^m / m)^{i_m}}{i_m!}.$$

Summing over $\mathbf{i}$ and m, we get

$$S(x, y) = \prod_m \exp(\sum_{j,k} x_j^m y_k^m / m) = \exp(-\sum_{j,k} \log(1 - x_j y_k)) = \prod_{j,k} (1 - x_j y_k)^{-1}$$

Thus,

$$R(x, y) = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (1 - x_i y_j)}.$$

Now we need the following lemma.

Lemma 4.48.

$$\frac{\prod_{i<j} (z_j - z_i)(y_i - y_j)}{\prod_{i,j} (z_i - y_j)} = \det\left(\frac{1}{z_i - y_j}\right).$$

Proof. Multiply both sides by $\prod_{i,j} (z_i - y_j)$. Then the right hand side must vanish on the hyperplanes $z_i = z_j$ and $y_i = y_j$ (i.e., be divisible by $\Delta(z)\Delta(y)$), and is a homogeneous polynomial of degree $N(N-1)$. This implies that the right hand side and the left hand side are proportional. The proportionality coefficient (which is equal to 1) is found by induction by multiplying both sides by $z_N - y_N$ and then setting $z_N = y_N$. □

Now setting in the lemma $z_i = 1/x_i$, we get

Corollary 4.49. *(Cauchy identity)*

$$R(x, y) = \det\left(\frac{1}{1 - x_i y_j}\right) = \sum_{\sigma \in S_N} \frac{1}{\prod_{j=1}^N (1 - x_j y_{\sigma(j)})}.$$

Corollary 4.49 easily implies that the coefficient of $x^{\lambda+\rho}y^{\lambda+\rho}$ is 1. Indeed, if $\sigma = 1$ is a permutation in S_N, the coefficient of this monomial in $\frac{1}{\prod(1-x_j y_{\sigma(j)})}$ is obviously zero. $\square$

Remark. For partitions λ and μ of n, let us say that $\lambda \preceq \mu$ or $\mu \succeq \lambda$ if $\mu - \lambda$ is a sum of vectors of the form $e_i - e_j$, $i < j$ (called positive roots). This is a partial order, and $\mu \succeq \lambda$ implies $\mu \geq \lambda$. It follows from Theorem 4.47 and its proof that

$$\chi_\lambda = \oplus_{\mu \succeq \lambda} \widetilde{K}_{\mu\lambda} \chi_{U_\mu}.$$

This implies that the Kostka numbers $K_{\mu\lambda}$ vanish unless $\mu \succeq \lambda$.

4.16 Problems

In the following problems, we do not make a distinction between Young diagrams and partitions.

Problem 4.50. *For a Young diagram μ, let $A(\mu)$ be the set of Young diagrams obtained by adding a square to μ, and $R(\mu)$ be the set of Young diagrams obtained by removing a square from μ.*

(a) Show that $\mathrm{Res}_{S_{n-1}}^{S_n} V_\mu = \oplus_{\lambda \in R(\mu)} V_\lambda$.

(b) Show that $\mathrm{Ind}_{S_{n-1}}^{S_n} V_\mu = \oplus_{\lambda \in A(\mu)} V_\lambda$.

Problem 4.51. *The content $c(\lambda)$ of a Young diagram λ is the sum $\sum_j \sum_{i=1}^{\lambda_j} (i - j)$. Let $C = \sum_{i<j} (ij) \in \mathbb{C}[S_n]$ be the sum of all transpositions. Show that C acts on the Specht module V_λ by multiplication by $c(\lambda)$.*

Problem 4.52. *(a) Let V be any finite dimensional representation of S_n. Show that the element $E := (12) + ... + (1n)$ is diagonalizable and has integer eigenvalues on V, which are between $1 - n$ and $n - 1$.*

Hint. Represent E as $C_n - C_{n-1}$, where $C_n = C$ is the element from Problem 4.51.

(b) Show that the element $(12) + ... + (1n)$ acts on V_λ by a scalar if and only if λ is a rectangular Young diagram, and compute this scalar.

4.17 The hook length formula

Let us use the Frobenius character formula to compute the dimension of V_λ. According to the character formula, $\dim V_\lambda$ is the coefficient of $x^{\lambda+\rho}$ in $\Delta(x)(x_1 + ... + x_N)^n$. Let $l_j = \lambda_j + N - j$. Then, using the determinant formula for $\Delta(x)$ and expanding the determinant as a sum over permutations, we get

$$\dim V_\lambda = \sum_{s \in S_N : l_j \geq N - s(j)} (-1)^s \frac{n!}{\prod_j (l_j - N + s(j))!} = \frac{n!}{l_1! ... l_N!} \sum_{s \in S_N} (-1)^s \prod_j l_j(l_j-1)...(l_j-N+s(j)+1) =$$

$$\frac{n!}{\prod_j l_j!} \det(l_j(l_j - 1)...(l_j - N + i + 1)).$$

Using column reduction and the Vandermonde determinant formula, we see from this expression that

$$\dim V_\lambda = \frac{n!}{\prod_j l_j!} \det(l_j^{N-i}) = \frac{n!}{\prod_j l_j!} \prod_{1 \leq i < j \leq N} (l_i - l_j) \qquad (5)$$

(where $N \geq p$).

In this formula, there are many cancelations. After making some of these cancelations, we obtain the hook length formula. Namely, for a square (i, j) in a Young diagram λ $(i, j \geq 1, i \leq \lambda_j)$, define the hook of (i, j) to be the set of all squares (i', j') in λ with $i' \geq i$, $j' = j$ or $i' = i$, $j' \geq j$. Let $h(i, j)$ be the length of the hook of i, j, i.e., the number of squares in it.

Theorem 4.53. *(The hook length formula) One has*

$$\dim V_\lambda = \frac{n!}{\prod_{i \leq \lambda_j} h(i,j)}.$$

Proof. The formula follows from formula (5). Namely, note that

$$\frac{l_1!}{\prod_{1<j\leq N}(l_1 - l_j)} = \prod_{1 \leq k \leq l_1, k \neq l_1 - l_j} k.$$

It is easy to see that the factors in this product are exactly the hooklengths $h(i, 1)$. Now delete the first row of the diagram and proceed by induction. $\square$

4.18 Schur-Weyl duality for $\mathfrak{gl}(V)$

We start with a simple result which is called the Double Centralizer Theorem.

Theorem 4.54. *Let A, B be two subalgebras of the algebra $\operatorname{End} E$ of endomorphisms of a finite dimensional vector space E, such that A is semisimple, and $B = \operatorname{End}_A E$. Then:*

(i) $A = \operatorname{End}_B E$ (i.e., the centralizer of the centralizer of A is A);

(ii) B is semisimple;

(iii) as a representation of $A \otimes B$, E decomposes as $E = \oplus_{i \in I} V_i \otimes W_i$, where V_i are all the irreducible representations of A, and W_i are all the irreducible representations of B. In particular, we have a natural bijection between irreducible representations of A and B.

Proof. Since A is semisimple, we have a natural decomposition $E = \oplus_{i \in I} V_i \otimes W_i$, where $W_i := \operatorname{Hom}_A(V_i, E)$, and $A = \oplus_i \operatorname{End} V_i$. Therefore, by Schur's lemma, $B = \operatorname{End}_A(E)$ is naturally identified with $\oplus_i \operatorname{End}(W_i)$. This implies all the statements of the theorem. $\square$

We will now apply Theorem 4.54 to the following situation: $E = V^{\otimes n}$, where V is a finite dimensional vector space over a field of characteristic zero, and A is the image of $\mathbb{C}[S_n]$ in $\operatorname{End} E$. Let us now characterize the algebra B. Let $\mathfrak{gl}(V)$ be $\operatorname{End} V$ regarded as a Lie algebra with operation $ab - ba$.

Theorem 4.55. *The algebra $B = \operatorname{End}_A E$ is the image of the universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}(V))$ under its natural action on E. In other words, B is generated by elements of the form*

$$\Delta_n(b) := b \otimes 1 \otimes ... \otimes 1 + 1 \otimes b \otimes ... \otimes 1 + ... + 1 \otimes 1 \otimes ... \otimes b,$$

$b \in \mathfrak{gl}(V)$.

Proof. Clearly, the image of $\mathfrak{U}(\mathfrak{gl}(V))$ is contained in B, so we just need to show that any element of B is contained in the image of $\mathfrak{U}(\mathfrak{gl}(V))$. By definition, $B = S^n \operatorname{End} V$, so the result follows from part (ii) of the following lemma.

Lemma 4.56. *Let k be a field of characteristic zero.*

(i) For any finite dimensional vector space U over k, the space S^nU is spanned by elements of the form $u \otimes ... \otimes u$, $u \in U$.

(ii) For any algebra A over k, the algebra S^nA is generated by elements $\Delta_n(a)$, $a \in A$.

Proof. (i) The space S^nU is an irreducible representation of $GL(U)$ (Problem 3.19). The subspace spanned by $u \otimes ... \otimes u$ is a nonzero subrepresentation, so it must be everything.

(ii) By the fundamental theorem on symmetric functions, there exists a polynomial P with rational coefficients such that $P(H_1(x), ..., H_n(x)) = x_1...x_n$ (where $x = (x_1, ..., x_n)$). Then

$$P(\Delta_n(a), \Delta_n(a^2), ..., \Delta_n(a^n)) = a \otimes ... \otimes a.$$

The rest follows from (i). $\square$

$\square$

Now, the algebra A is semisimple by Maschke's theorem, so the double centralizer theorem applies, and we get the following result, which goes under the name "Schur-Weyl duality".

Theorem 4.57. *(i) The image A of $\mathbb{C}[S_n]$ and the image B of $\mathfrak{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ are centralizers of each other.*

(ii) Both A and B are semisimple. In particular, $V^{\otimes n}$ is a semisimple $\mathfrak{gl}(V)$-module.

(iii) We have a decomposition of $A \otimes B$-modules $V^{\otimes n} = \oplus_\lambda V_\lambda \otimes L_\lambda$, where the summation is taken over partitions of n, V_λ are Specht modules for S_n, and L_λ are some distinct irreducible representations of $\mathfrak{gl}(V)$ or zero.

4.19 Schur-Weyl duality for $GL(V)$

The Schur-Weyl duality for the Lie algebra $\mathfrak{gl}(V)$ implies a similar statement for the group $GL(V)$.

Proposition 4.58. *The image of $GL(V)$ in $\mathrm{End}(V^{\otimes n})$ spans B.*

Proof. Denote the span of $g^{\otimes n}$, $g \in GL(V)$, by B'. Let $b \in \mathrm{End}\, V$ be any element.

We claim that B' contains $b^{\otimes n}$. Indeed, for all values of t but finitely many, $t \cdot \mathrm{Id} + b$ is invertible, so $(t \cdot \mathrm{Id} + b)^{\otimes n}$ belongs to B'. This implies that this is true for all t, in particular for $t = 0$, since $(t \cdot \mathrm{Id} + b)^{\otimes n}$ is a polynomial in t.

The rest follows from Lemma 4.56. $\square$

Corollary 4.59. *As a representation of $S_n \times GL(V)$, $V^{\otimes n}$ decomposes as $\oplus_\lambda V_\lambda \otimes L_\lambda$, where $L_\lambda = Hom_{S_n}(V_\lambda, V^{\otimes n})$ are distinct irreducible representations of $GL(V)$ or zero.*

Example 4.60. If $\lambda = (n)$ then $L_\lambda = S^nV$, and if $\lambda = (1^n)$ (n copies of 1) then $L_\lambda = \wedge^n V$. It was shown in Problem 3.19 that these representations are indeed irreducible (except that $\wedge^n V$ is zero if $n > \dim V$).

4.20 Schur polynomials

Let $\lambda = (\lambda_1, ..., \lambda_p)$ be a partition of n, and $N \geq p$. Let

$$D_\lambda(x) = \sum_{s \in S_N} (-1)^s \prod_{j=1}^{N} x_{s(j)}^{\lambda_j + N - j} = \det(x_i^{\lambda_j + N - j}).$$

Define the polynomials

$$S_\lambda(x) := \frac{D_\lambda(x)}{D_0(x)}$$

(clearly $D_0(x)$ is just $\Delta(x)$). It is easy to see that these are indeed polynomials, as D_λ is antisymmetric and therefore must be divisible by Δ. The polynomials S_λ are called the **Schur polynomials**.

Proposition 4.61.

$$\prod_m (x_1^m + ... + x_N^m)^{i_m} = \sum_{\lambda : p \leq N} \chi_\lambda(C_{\mathbf{i}}) S_\lambda(x).$$

Proof. The identity follows from the Frobenius character formula and the antisymmetry of

$$\Delta(x) \prod_m (x_1^m + ... + x_N^m)^{i_m}.$$

□

Certain special values of Schur polynomials are of importance. Namely, we have

Proposition 4.62.

$$S_\lambda(1, z, z^2, ..., z^{N-1}) = \prod_{1 \leq i < j \leq N} \frac{z^{\lambda_i - i} - z^{\lambda_j - j}}{z^{-i} - z^{-j}}$$

Therefore,

$$S_\lambda(1, ..., 1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Proof. The first identity is obtained from the definition using the Vandermonde determinant. The second identity follows from the first one by setting $z = 1$. □

4.21 The characters of L_λ

Proposition 4.61 allows us to calculate the characters of the representations L_λ.

Namely, let $\dim V = N$, $g \in GL(V)$, and $x_1, ..., x_N$ be the eigenvalues of g on V. To compute the character $\chi_{L_\lambda}(g)$, let us calculate $\mathrm{Tr}_{V^{\otimes n}}(g^{\otimes n} s)$, where $s \in S_n$. If $s \in C_{\mathbf{i}}$, we easily get that this trace equals

$$\prod_m \mathrm{Tr}(g^m)^{i_m} = \prod_m H_m(x)^{i_m}.$$

On the other hand, by the Schur-Weyl duality

$$\mathrm{Tr}_{V^{\otimes n}}(g^{\otimes n} s) = \sum_\lambda \chi_\lambda(C_{\mathbf{i}}) \mathrm{Tr}_{L_\lambda}(g).$$

Comparing this to Proposition 4.61 and using linear independence of columns of the character table of S_n, we obtain

Theorem 4.63. *(Weyl character formula) The representation L_λ is zero if and only if $N < p$, where p is the number of parts of λ. If $N \geq p$, the character of L_λ is the Schur polynomial $S_\lambda(x)$. Therefore, the dimension of L_λ is given by the formula*

$$\dim L_\lambda = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

This shows that irreducible representations of $GL(V)$ which occur in $V^{\otimes n}$ for some n are labeled by Young diagrams with any number of squares but at most $N = \dim V$ rows.

Proposition 4.64. *The representation $L_{\lambda+1^N}$ (where $1^N = (1, 1, ..., 1) \in \mathbb{Z}^N$) is isomorphic to $L_\lambda \otimes \wedge^N V$.*

Proof. Indeed, $L_\lambda \otimes \wedge^N V \subset V^{\otimes n} \otimes \wedge^N V \subset V^{\otimes n+N}$, and the only component of $V^{\otimes n+N}$ that has the same character as $L_\lambda \otimes \wedge^N V$ is $L_{\lambda+1^N}$. This implies the statement. □

4.22 Polynomial representations of $GL(V)$

Definition 4.65. We say that a finite dimensional representation Y of $GL(V)$ is **polynomial** (or **algebraic**, or **rational**) if its matrix elements are polynomial functions of the entries of g, g^{-1}, $g \in GL(V)$ (i.e., belong to $k[g_{ij}][1/\det(g)]$).

For example, $V^{\otimes n}$ and hence all L_λ are polynomial. Also define $L_{\lambda - r \cdot 1^N} := L_\lambda \otimes (\wedge^N V^*)^{\otimes r}$ (this definition makes sense by Proposition 4.64). This is also a polynomial representation. Thus we have attached a unique irreducible polynomial representation L_λ of $GL(V) = GL_N$ to any sequence $(\lambda_1, ..., \lambda_N)$ of integers (not necessarily positive) such that $\lambda_1 \geq ... \geq \lambda_N$. This sequence is called the **highest weight** of L_λ.

Theorem 4.66. *(i) Every finite dimensional polynomial representation of $GL(V)$ is completely reducible, and decomposes into summands of the form L_λ (which are pairwise non-isomorphic).*

(ii) (the Peter-Weyl theorem for $GL(V)$). Let R be the algebra of polynomial functions on $GL(V)$. Then as a representation of $GL(V) \times GL(V)$ (with action $(\rho(g, h)\phi)(x) = \phi(g^{-1}xh)$, $g, h, x \in GL(V)$, $\phi \in R$), R decomposes as

$$R = \oplus_\lambda L_\lambda^* \otimes L_\lambda,$$

where the summation runs over all λ.

Proof. (i) Let Y be a polynomial representation of $GL(V)$. We have an embedding $\xi : Y \to Y \otimes R$ given by $(u, \xi(v))(g) := u(gv)$, $u \in V^*$. It is easy to see that ξ is a homomorphism of representations (where the action of $GL(V)$ on the first component of $Y \otimes R$ is trivial). Thus, it suffices to prove the theorem for a subrepresentation $Y \subset R^m$. Now, every element of R is a polynomial of g_{ij} times a nonpositive power of $\det(g)$. Thus, R is a quotient of a direct sum of representations of the form $S^r(V \otimes V^*) \otimes (\wedge^N V^*)^{\otimes s}$. So we may assume that Y is contained in a quotient of a (finite) direct sum of such representations. As $V^* = \wedge^{N-1} V \otimes \wedge^N V^*$, Y is contained in a direct sum of representations of the form $V^{\otimes n} \otimes (\wedge^N V^*)^{\otimes s}$, and we are done.

(ii) Let Y be a polynomial representation of $GL(V)$, and let us regard R as a representation of $GL(V)$ via $(\rho(h)\phi)(x) = \phi(xh)$. Then $\mathrm{Hom}_{GL(V)}(Y, R)$ is the space of polynomial functions on $GL(V)$ with values in Y^*, which are $GL(V)$-equivariant. This space is naturally identified with Y^*. Taking into account the proof of (i), we deduce that R has the required decomposition, which is compatible with the second action of $GL(V)$ (by left multiplications). This implies the statement. $\square$

Note that the Peter-Weyl theorem generalizes Maschke's theorem for finite group, one of whose forms states that the space of complex functions $\mathrm{Fun}(G, \mathbb{C})$ on a finite group G as a representation of $G \times G$ decomposes as $\oplus_{V \in \mathrm{Irrep(G)}} V^* \otimes V$.

Remark 4.67. Since the Lie algebra $\mathfrak{sl}(V)$ of traceless operators on V is a quotient of $\mathfrak{gl}(V)$ by scalars, the above results extend in a straightforward manner to representations of the Lie algebra $\mathfrak{sl}(V)$. Similarly, the results for $GL(V)$ extend to the case of the group $SL(V)$ of operators with determinant 1. The only difference is that in this case the representations L_λ and $L_{\lambda+1^m}$ are isomorphic, so the irreducible representations are parametrized by integer sequences $\lambda_1 \geq ... \geq \lambda_N$ up to a simultaneous shift by a constant.

In particular, one can show that any finite dimensional representation of $\mathfrak{sl}(V)$ is completely reducible, and any irreducible one is of the form L_λ (we will not do this here). For $\dim V = 2$ one then recovers the representation theory of $\mathfrak{sl}(2)$ studied in Problem 1.55.

4.23 Problems

Problem 4.68. *(a) Show that the S_n-representation $V'_\lambda := \mathbb{C}[S_n]b_\lambda a_\lambda$ is isomorphic to V_λ.*

Hint. Define S_n-homomorphisms $f : V_\lambda \to V'_\lambda$ and $g : V'_\lambda \to V_\lambda$ by the formulas $f(x) = xa_\lambda$ and $g(y) = yb_\lambda$, and show that they are inverse to each other up to a nonzero scalar.

(b) Let $\phi : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ be the automorphism sending s to $(-1)^s s$ for any permutation s. Show that ϕ maps any representation V of S_n to $V \otimes \mathbb{C}_-$. Show also that $\phi(\mathbb{C}[S_n]a) = \mathbb{C}[S_n]\phi(a)$, for $a \in \mathbb{C}[S_n]$. Use (a) to deduce that $V_\lambda \otimes \mathbb{C}_- = V_{\lambda^*}$, where λ^* is the conjugate partition to λ, obtained by reflecting the Young diagram of λ.

Problem 4.69. *Let $R_{k,N}$ be the algebra of polynomials on the space of k-tuples of complex N by N matrices $X_1, ..., X_k$, invariant under simultaneous conjugation. An example of an element of $R_{k,N}$ is the function $T_w := \mathrm{Tr}(w(X_1, ..., X_k))$, where w is any finite word on a k-letter alphabet. Show that $R_{k,N}$ is generated by the elements T_w.*

Hint. Consider invariant functions that are of degree d_i in each X_i, and realize this space as a tensor product $\otimes_i S^{d_i}(V \otimes V^*)$. Then embed this tensor product into $(V \otimes V^*)^{\otimes N} = \mathrm{End}(V)^{\otimes n}$, and use the Schur-Weyl duality to get the result.

4.24 Representations of $GL_2(\mathbb{F}_q)$

4.24.1 Conjugacy classes in $GL_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of size q of characteristic other than 2, and $G = GL_2(\mathbb{F}_q)$. Then

$$|G| = (q^2 - 1)(q^2 - q),$$

since the first column of an invertible 2 by 2 matrix must be non-zero and the second column may not be a multiple of the first one. Factoring,

$$|GL_2(\mathbb{F}_q)| = q(q + 1)(q - 1)^2.$$

The goal of this section is to describe the irreducible representations of G.
To begin, let us find the conjugacy classes in $GL_2(\mathbb{F}_q)$.

Representatives	Number of elements in a conjugacy class	Number of classes
Scalar $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	1 (this is a central element)	$q-1$ (one for every non-zero x)
Parabolic $\left(\begin{smallmatrix} x & 1 \\ 0 & x \end{smallmatrix}\right)$	q^2-1 (elements that commute with this one are of the form $\left(\begin{smallmatrix} t & u \\ 0 & t \end{smallmatrix}\right)$, $t =$ 0)	$q-1$ (one for every non-zero x)
Hyperbolic $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $y \neq x$	q^2+q (elements that commute with this one are of the form $\left(\begin{smallmatrix} t & 0 \\ 0 & u \end{smallmatrix}\right)$, $t, u =$ 0)	$\frac{1}{2}(q-1)(q-2)$ ($x, y = 0$ and $x \neq y$)
Elliptic $\left(\begin{smallmatrix} x & \varepsilon y \\ y & x \end{smallmatrix}\right), x \in \mathbb{F}_q,\ y \in \mathbb{F}_q^{\times},\ \varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$ (characteristic polynomial over $\mathbb{F}_q$ is irreducible)	q^2-q (the reason will be described below)	$\frac{1}{2}q(q-1)$ (matrices with y and $-y$ are conjugate)

More on the conjugacy class of elliptic matrices: these are the matrices whose characteristic polynomial is irreducible over $\mathbb{F}_q$ and which therefore don't have eigenvalues in $\mathbb{F}_q$. Let A be such a matrix, and consider a quadratic extension of $\mathbb{F}_q$,

$$\mathbb{F}_q(\sqrt{\varepsilon}), \varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2.$$

Over this field, A will have eigenvalues

$$\alpha = \alpha_1 + \sqrt{\varepsilon}\alpha_2$$

and

$$\overline{\alpha} = \alpha_1 - \sqrt{\varepsilon}\alpha_2,$$

with corresponding eigenvectors

$$v,\ \overline{v} \quad (Av = \alpha v,\ A\overline{v} = \overline{\alpha v}).$$

Choose a basis

$$\{e_1 = v + \overline{v},\ e_2 = \sqrt{\varepsilon}(v - \overline{v})\}.$$

In this basis, the matrix A will have the form

$$\begin{pmatrix} \alpha_1 & \varepsilon\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

justifying the description of representative elements of this conjugacy class.
In the basis $\{v, \overline{v}\}$, matrices that commute with A will have the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

for all

$$\lambda \in \mathbb{F}_{q^2}^{\times},$$

so the number of such matrices is $q^2 - 1$.

4.24.2 1-dimensional representations

First, we describe the 1-dimensional representations of G.

Proposition 4.70. $[G, G] = SL_2(\mathbb{F}_q)$.

Proof. Clearly,

$$\det(xyx^{-1}y^{-1}) = 1,$$

so

$$[G, G] \subseteq SL_2(\mathbb{F}_q).$$

To show the converse, it suffices to show that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are commutators (as such matrices generate $SL_2(\mathbb{F}_q)$.) Clearly, by using transposition, it suffices to show that only the first two matrices are commutators. But it is easy to see that the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is the commutator of the matrices

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

while the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

is the commutator of the matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

This completes the proof. ∎

Therefore,

$$G/[G, G] \cong \mathbb{F}_q^\times \text{ via } g \to \det(g).$$

The one-dimensional representations of G thus have the form

$$\rho(g) = \xi\big(\det(g)\big),$$

where ξ is a homomorphism

$$\xi : \mathbb{F}_q^\times \to \mathbb{C}^\times;$$

so there are $q - 1$ such representations, denoted $\mathbb{C}_\xi$.

4.24.3 Principal series representations

Let

$$B \subset G,\ B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

(the set of upper triangular matrices); then

$$|B| = (q-1)^2 q,$$

$$[B,B] = U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\},$$

and

$$B/[B,B] \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

(the isomorphism maps an element of B to its two diagonal entries).
Let

$$\lambda : B \to \mathbb{C}^\times$$

be a homomorphism defined by

$$\lambda \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \lambda_1(a)\lambda_2(c), \text{for some pair of homomorphisms } \lambda_1, \lambda_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times.$$

Define

$$V_{\lambda_1,\lambda_2} = \mathrm{Ind}_B^G \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional representation of B in which B acts by λ. We have

$$\dim(V_{\lambda_1,\lambda_2}) = \frac{|G|}{|B|} = q+1.$$

Theorem 4.71. *1. $\lambda_1 \neq \lambda_2 \Rightarrow V_{\lambda_1,\lambda_2}$ is irreducible.*

2. $\lambda_1 = \lambda_2 = \mu \Rightarrow V_{\lambda_1,\lambda_2} = \mathbb{C}_\mu \oplus W_\mu$, where W_μ is a q-dimensional irreducible representation of G.

3. $W_\mu \cong W_\nu$ if and only if $\mu = \nu$; $V_{\lambda_1,\lambda_2} \cong V_{\lambda_1',\lambda_2'}$ if and only if $\{\lambda_1,\lambda_2\} = \{\lambda_1',\lambda_2'\}$ (in the second case, $\lambda_1 \neq \lambda_2, \lambda_1' \neq \lambda_2'$).

Proof. From the Mackey formula, we have

$$tr_{V_{\lambda_1,\lambda_2}}(g) = \frac{1}{|B|} \sum_{a\in G,\, aga^{-1}\in B} \lambda(aga^{-1}).$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

the expression on the right evaluates to

$$\lambda(g)\frac{|G|}{|B|} = \lambda_1(x)\lambda_2(x)\big(q+1\big).$$

If

$$g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix},$$

the expression evaluates to

$$\lambda(g) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B.$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

the expression evaluates to

$$\big(\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)\big) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B \text{ or } a \text{ is an element of } B \text{ multiplied by the transposition matrix.}$$

If

$$g = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, \; x \neq y$$

the expression on the right evaluates to 0 because matrices of this type don't have eigenvalues over $\mathbb{F}_q$ (and thus cannot be conjugated into B). From the definition, $\lambda_i(x)(i = 1, 2)$ is a root of unity, so

$$\begin{aligned} |G|\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}} \rangle &= (q+1)^2(q-1) + (q^2-1)(q-1) \\ &\quad + 2(q^2+q)\frac{(q-1)(q-2)}{2} + (q^2+q)\sum_{x=y} \lambda_1(x)\lambda_2(y)\overline{\lambda_1(y)\lambda_2(x)}. \end{aligned}$$

The last two summands come from the expansion

$$|a+b|^2 = |a|^2 + |b|^2 + a\overline{b} + \overline{a}b.$$

If

$$\lambda_1 = \lambda_2 = \mu,$$

the last term is equal to

$$(q^2+q)(q-2)(q-1),$$

and the total in this case is

$$(q+1)(q-1)[(q+1) + (q-1) + 2q(q-2)] = (q+1)(q-1)2q(q-1) = 2|G|,$$

so

$$\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}} \rangle = 2.$$

Clearly,

$$\mathbb{C}_\mu \subseteq \mathrm{Ind}_B^G \mathbb{C}_{\mu,\mu},$$

since

$$\mathrm{Hom}_G(\mathbb{C}_\mu, \mathrm{Ind}_B^G \mathbb{C}_{\mu,\mu}) = \mathrm{Hom}_B(\mathbb{C}_\mu, \mathbb{C}_\mu) = \mathbb{C} \text{ (Theorem 4.33).}$$

Therefore, $\mathrm{Ind}_B^G \mathbb{C}_{\mu,\mu} = \mathbb{C}_\mu \oplus W_\mu$; W_μ is irreducible; and the character of W_μ is different for distinct values of μ, proving that W_μ are distinct.

If $\lambda_1 \neq \lambda_2$, let $z = xy^{-1}$, then the last term of the summation is

$$(q^2+q)\sum_{x=y} \lambda_1(z)\overline{\lambda_2(z)} = (q^2+q)\sum_{x;z=1} \frac{\lambda_1}{\lambda_2}(z) = (q^2+q)(q-1)\sum_{z=1} \frac{\lambda_1}{\lambda_2}(z).$$

Since

$$\sum_{z\in\mathbb{F}_q^\times} \frac{\lambda_1}{\lambda_2}(z) = 0,$$

because the sum of all roots of unity of a given order $m > 1$ is zero, the last term becomes

$$-(q^2+q)(q-1)\sum_{z=1} \frac{\lambda_1}{\lambda_2}(1) = -(q^2+q)(q-1).$$

The difference between this case and the case of $\lambda_1 = \lambda_2$ is equal to

$$-(q^2+q)[(q-2)(q-1)+(q-1)] = |G|,$$

so this is an irreducible representation by Lemma 4.27.

To prove the third assertion of the theorem, we look at the characters on hyperbolic elements and note that the function

$$\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)$$

determines λ_1, λ_2 up to permutation.

$\square$

4.24.4 Complementary series representations

Let $\mathbb{F}_{q^2} \supset \mathbb{F}_q$ be a quadratic extension $\mathbb{F}_q(\sqrt{\varepsilon}), \varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$. We regard this as a 2-dimensional vector space over $\mathbb{F}_q$; then G is the group of linear transformations of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Let $K \subset G$ be the cyclic group of multiplications by elements of $\mathbb{F}_{q^2}^\times$,

$$K = \{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}\},\ |K| = q^2-1.$$

For $\nu : K \to \mathbb{C}^\times$ a homomorphism, let

$$Y_\nu = \mathrm{Ind}_K^G \mathbb{C}_\nu.$$

This representation, of course, is very reducible. Let us compute its character, using the Mackey formula. We get

$$\chi\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q-1)\nu(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or hyperbolic;}$$

$$\chi\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \nu\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} + \nu\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}^q.$$

The last assertion holds because if we regard the matrix as an element of $\mathbb{F}_{q^2}$, conjugation is an automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, but the only nontrivial automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ is the q^{th} power map.

We thus have

$$\mathrm{Ind}_K^G \mathbb{C}_{\nu^q} \cong \mathrm{Ind}_K^G \mathbb{C}_\nu$$

because they have the same character. Therefore, for $\nu^q \neq \nu$ we get $\frac{1}{2}q(q-1)$ representations.

Next, we look at the following tensor product:

$$W_1 \otimes V_{\alpha,1},$$

where 1 is the trivial character and W_1 is defined as in the previous section. The character of this representation is

$$\chi\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q+1)\alpha(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or elliptic};$$

$$\chi\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \alpha(x) + \alpha(y).$$

Thus the "virtual representation"

$$W_1 \otimes V_{\alpha,1} - V_{\alpha,1} - \mathrm{Ind}_K^G \mathbb{C}_\nu,$$

where α is the restriction of ν to scalars, has character

$$\chi\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = (q-1)\alpha(x);$$

$$\chi\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = -\alpha(x);$$

$$\chi\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = 0;$$

$$\chi\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = -\nu\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} - \nu^q\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}.$$

In all that follows, we will have $\nu^q \neq \nu$.

The following two lemmas will establish that the inner product of this character with itself is equal to 1, that its value at 1 is positive. As we know from Lemma 4.27, these two properties imply that it is the character of an irreducible representation of G.

Lemma 4.72. *Let χ be the character of the "virtual representation" defined above. Then*

$$\langle \chi, \chi \rangle = 1$$

and

$$\chi(1) > 0.$$

Proof.

$$\chi(1) = q(q+1) - (q+1) - q(q-1) = q - 1 > 0.$$

We now compute the inner product $\langle \chi, \chi \rangle$. Since α is a root of unity, this will be equal to

$$\frac{1}{(q-1)^2 q(q+1)}\Big[(q-1)\cdot(q-1)^2\cdot 1 + (q-1)\cdot 1\cdot(q^2-1) + \frac{q(q-1)}{2}\cdot \sum_{\zeta \text{ elliptic}} (\nu(\zeta)+\nu^q(\zeta))\overline{(\nu(\zeta)+\nu^q(\zeta))}\Big]$$

Because ν is also a root of unity, the last term of the expression evaluates to

$$\sum_{\zeta \text{ elliptic}} (2 + \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta)).$$

Let's evaluate the last summand.

Since $\mathbb{F}_{q^2}^\times$ is cyclic and $\nu^q \neq \nu$,

$$\sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{q-1}(\zeta) = \sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{1-q}(\zeta) = 0.$$

Therefore,

$$\sum_{\zeta \text{ elliptic}} (\nu^{q-1}(\zeta) + \nu^{1-q}(\zeta)) = -\sum_{\zeta \in \mathbb{F}_q^\times} (\nu^{q-1}(\zeta) + \nu^{1-q}(\zeta)) = -2(q-1) =$$

since $\mathbb{F}_q^\times$ is cyclic of order $q-1$. Therefore,

$$\langle \chi, \chi \rangle = \frac{1}{(q-1)^2 q(q+1)} \left((q-1)\cdot(q-1)^2\cdot 1 + (q-1)\cdot 1\cdot(q^2-1) + \frac{q(q-1)}{2}\cdot(2(q^2-q) - 2(q-1)) \right) = 1.$$

$\square$

We have now shown that for any ν with $\nu^q \neq \nu$ the representation Y_ν with the same character as

$$W_1 \otimes V_{\alpha,1} - V_{\alpha,1} - \mathrm{Ind}_K^G \mathbb{C}_\nu$$

exists and is irreducible. These characters are distinct for distinct pairs $(\alpha, \ \nu)$ (up to switch $\nu \to \nu^q$), so there are $\frac{q(q-1)}{2}$ such representations, each of dimension $q-1$.

We have thus found $q-1$ 1-dimensional representations of G, $\frac{q(q-1)}{2}$ principal series representations, and $\frac{q(q-1)}{2}$ complementary series representations, for a total of q^2-1 representations, i.e., the number of conjugacy classes in G. This implies that we have in fact found all irreducible representations of $GL_2(\mathbb{F}_q)$.

4.25 Artin's theorem

Theorem 4.73. *Let X be a conjugation-invariant system of subgroups of a finite group G. Then two conditions are equivalent:*

(i) Any element of G belongs to a subgroup $H \in X$.

(ii) The character of any irreducible representation of G belongs to the $\mathbb{Q}$-span of characters of induced representations $\mathrm{Ind}_H^G V$, where $H \in X$ and V is an irreducible representation of H.

Remark. Statement (ii) of Theorem 4.73 is equivalent to the same statement with $\mathbb{Q}$-span replaced by $\mathbb{C}$-span. Indeed, consider the matrix whose columns consist of the coefficients of the decomposition of $\mathrm{Ind}_H^G V$ (for various H, V) with respect to the irreducible representations of G. Then both statements are equivalent to the condition that the rows of this matrix are linearly independent.

Proof. Proof that (ii) implies (i). Assume that $g \in G$ does not belong to any of the subgroups $H \in X$. Then, since X is conjugation invariant, it cannot be conjugated into such a subgroup. Hence by the Mackey formula, $\chi_{\mathrm{Ind}_H^G(V)}(g) = 0$ for all $H \in X$ and V. So by (ii), for any irreducible representation W of G, $\chi_W(g) = 0$. But irreducible characters span the space of class functions, so any class function vanishes on g, which is a contradiction.

Proof that (i) implies (ii). Let U be a virtual representation of G over $\mathbb{C}$ (i.e., a linear combination of irreducible representations with nonzero integer coefficients) such that $(\chi_U, \chi_{\mathrm{Ind}_H^G V}) = 0$ for all H, V. So by Frobenius reciprocity, $(\chi_{U|_H}, \chi_V) = 0$. This means that χ_U vanishes on H for any $H \in X$. Hence by (i), χ_U is identically zero. This implies (ii) (because of the above remark). $\square$

Corollary 4.74. *Any irreducible character of a finite group is a rational linear combination of induced characters from its cyclic subgroups.*

4.26 Representations of semidirect products

Let G, A be groups and $\phi : G \to \mathrm{Aut}(A)$ be a homomorphism. For $a \in A$, denote $\phi(g)a$ by $g(a)$. The semidirect product $G \ltimes A$ is defined to be the product $A \times G$ with multiplication law

$$(a_1, g_1)(a_2, g_2) = (a_1 g_1(a_2), g_1 g_2).$$

Clearly, G and A are subgroups of $G \ltimes A$ in a natural way.

We would like to study irreducible complex representations of $G \ltimes A$. For simplicity, let us do it when A is abelian.

In this case, irreducible representations of A are 1-dimensional and form the character group $A^\vee$, which carries an action of G. Let O be an orbit of this action, $x \in O$ a chosen element, and G_x the stabilizer of x in G. Let U be an irreducible representation of G_x. Then we define a representation $V_{(O,U)}$ of $G \ltimes A$ as follows.

As a representation of G, we set

$$V_{(O,x,U)} = \mathrm{Ind}_{G_x}^G U = \{f : G \to U | f(hg) = hf(g), h \in G_x\}.$$

Next, we introduce an additional action of A on this space by $(af)(g) = x(g(a))f(g)$. Then it's easy to check that these two actions combine into an action of $G \ltimes A$. Also, it is clear that this representation does not really depend on the choice of x, in the following sense. Let $x, y \in O$, and $g \in G$ be such that $gx = y$, and let $g(U)$ be the representation of G_y obtained from the representation U of G_x by the action of g. Then $V_{(O,x,U)}$ is (naturally) isomorphic to $V_{(O,y,g(U))}$. Thus we will denote $V_{(O,x,U)}$ by $V_{(O,U)}$ (remembering, however, that x has been fixed).

Theorem 4.75. *(i) The representations $V_{(O,U)}$ are irreducible.*

(ii) They are pairwise nonisomorphic.

(iii) They form a complete set of irreducible representations of $G \ltimes A$.

(iv) The character of $V = V_{(O,U)}$ is given by the Mackey-type formula

$$\chi_V(a, g) = \frac{1}{|G_x|} \sum_{h \in G : hgh^{-1} \in G_x} x(h(a)) \chi_U(hgh^{-1}).$$

Proof. (i) Let us decompose $V = V_{(O,U)}$ as an A-module. Then we get

$$V = \oplus_{y\in O} V_y,$$

where $V_y = \{v \in V_{(O,U)} | av = (y,a)v, a \in A\}$. (Equivalently, $V_y = \{v \in V_{(O,U)} | v(g) = 0$ unless $gy = x\}$). So if $W \subset V$ is a subrepresentation, then $W = \oplus_{y\in O} W_y$, where $W_y \subset V_y$. Now, V_y is a representation of G_y, which goes to U under any isomorphism $G_y \to G_x$ determined by $g \in G$ mapping x to y. Hence, V_y is irreducible over G_y, so $W_y = 0$ or $W_y = V_y$ for each y. Also, if $hy = z$ then $hW_y = W_z$, so either $W_y = 0$ for all y or $W_y = V_y$ for all y, as desired.

(ii) The orbit O is determined by the A-module structure of V, and the representation U by the structure of V_x as a G_x-module.

(iii) We have

$$\sum_{U,O} \dim V^2_{(U,O)} = \sum_{U,O} |O|^2 (\dim U)^2 =$$

$$\sum_O |O|^2 |G_x| = \sum_O |O||G/G_x||G_x| = |G| \sum_O |O| = |G||A^\vee| = |G \ltimes A|.$$

(iv) The proof is essentially the same as that of the Mackey formula. $\square$

Exercise. Redo Problems 3.17(a), 3.18, 3.22 using Theorem 4.75.

Exercise. Deduce parts (i)-(iii) of Theorem 4.75 from part (iv).

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