

# 18.445 Introduction to Stochastic Processes

## Lecture 2: Markov chains: stationary distribution

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## Announcement : The midterm is postponed to April 6th.

### Recall

Consider a Markov chain with state space  $\Omega$  and transition matrix  $P$  :

$$\mathbb{P}[X_{n+1} = y \mid X_n = x] = P(x, y).$$

- $\mu_0$  : the distribution of  $X_0$  ;  $\mu_n$  : the distribution of  $X_n$ .
- We call a probability measure  $\pi$  is stationary if  $\pi = \pi P$ .
- If  $\pi$  is stationary and  $\mu_0 = \pi$ , then  $\mu_n = \pi, \forall n$ .

### Today's goal

- Irreducible, aperiodic
- existence and uniqueness of stationary distribution.

# Irreducible

## Definition

A transition matrix  $P$  is called irreducible, if for any  $x, y \in \Omega$ , there exists a number  $n$  (possibly depending on  $x, y$ ) such that

$$P^n(x, y) > 0.$$

## Definition

For any  $x \in \Omega$ , define  $T(x) = \{n \geq 1 : P^n(x, x) > 0\}$ . The period of state  $x$  is the greatest common divisor of  $T(x)$ , denoted by  $\gcd(T(x))$ .

## Lemma

If  $P$  is irreducible, then  $\gcd(T(x)) = \gcd(T(y))$  for all  $x, y \in \Omega$ .

# Aperiodic

## Definition

For an irreducible chain, the period of the chain is defined to be the period which is common to all states.

The chain is aperiodic if all states have period 1.

**Example** Consider a simple random walk on  $N$ -cycle where  $N$  is odd. Then the walk is irreducible and aperiodic.

## Theorem

*If  $P$  is irreducible and aperiodic, then there exists an integer  $r$  such that*

$$P^n(x, y) > 0, \quad \forall x, y \in \Omega, \forall n \geq r.$$

# Existence of stationary distribution

## Definition

For  $x \in \Omega$ , define

$$\tau_x = \inf\{n \geq 0 : X_n = x\}, \quad \tau_x^+ = \inf\{n \geq 1 : X_n = x\}.$$

$\tau_x$  : the hitting time for  $x$ .  $\tau_x^+$  : the first return time when  $X_0 = x$ .

## Lemma

*Suppose that  $P$  is irreducible. Then, for any  $x, y \in \Omega$ , we have*

$$\mathbb{E}_x[\tau_y^+] < \infty.$$

## Theorem

*Suppose that  $P$  is irreducible, then there exists a probability measure  $\pi$  such that  $\pi = \pi P$  and  $\pi(x) > 0$  for all  $x \in \Omega$ .*

# Uniqueness of stationary distribution

## Recall and Definition

- $\mu$  : a measure on  $\Omega$ ,  $\mu P$  is still a measure on  $\Omega$ .
- $f$  : a function on  $\Omega$ ,  $Pf$  is still a function on  $\Omega$ .
- If  $\mu = \mu P$ , we say that  $\mu$  is stationary.
- If  $f = Pf$ , we say that  $f$  is harmonic.

## Lemma

*Suppose that  $P$  is irreducible. Then any harmonic function  $f$  on  $\Omega$  has to be constant.*

## Theorem

*Suppose that  $P$  is irreducible. Then there exists a unique stationary distribution. Moreover,*

$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]}, \quad \forall x \in \Omega.$$

# Summary about stationary distribution

- In the proof of the existence and the uniqueness of stationary distribution, the crucial assumption is that  $P$  is irreducible.
- When  $P$  is irreducible, we can explicitly write out the stationary distribution

$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]} > 0, \forall x \in \Omega.$$

- In fact, the existence does not require irreducibility, but the uniqueness does.

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