**18.433 Combinatorial Optimization** Due: September 30

# Assignment 1: Matching Theory

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1. Show that a graph is bipartite if and only if it has no odd length cycles.

2. Consider the following *greedy* algorithm for finding a maximum matching: Start with an arbitrary edge as the initial matching. Find another edge that does not have a vertex in common with the current matching. If one exists, add it to the current matching. Repeat till no more edges can be added.

   (a) What is the running time of this algorithm on a graph with $n$ vertices and $m$ edges?

   (b) Give an example of a graph where the algorithm does not find the maximum matching.

   (c) Show that the matching found by the algorithm always has at least half as many edges as a maximum matching.

3. Now consider a similar algorithm for finding a *maximum weight* matching in an edge-weighted graph: Greedily add the heaviest edge possible to the current matching; stop when no further edge can be added. Show that this algorithm finds a matching whose weight is at least half the optimum.

4. A graph property $P$ (e.g. bipartiteness, existence of a perfect matching, existence of a Hamiltonian cycle) is said to have a *good characterization* if the truth or falsehood of the property can be *efficiently* verified. By this we mean that if the graph has $n$ vertices, then the proof of the property being true or false has length at most a polynomial in $n$. We saw in class that for a bipartite graph $G$, the property "$G$ has a perfect matching" has a good characterization. Which of the following properties have good characterizations? (prove your answer)

   (a) G is connected.

   (b) G is Eulerian, i.e., there exists a tour of $G$ that visits every edge exactly once.

   (c) G is 2-colorable, i.e., the vertices of $G$ can each be assigned one of two colors so that the endpoints of every edge have different colors.

5. Recall the Hungarian algorithm for finding a maximum matching in a bipartite graph. For finding alternating paths, the algorithm uses an alternating forest.

Additionally, the improved version of the algorithm, for each augmenting path finding phase, augments on a maximal set of disjoint augmenting paths. Prove that the length of the shortest augmenting path increases in each phase.

6. Let $G = (V, E)$ be a graph without isolated vertices. Recall that a *vertex cover* of $G$ is a set $U \subseteq V$ such that every edge of $G$ is incident to a vertex in $U$, and an *independent set* of $G$ is a set $U \subseteq V$ such that no edge of $G$ has both extremes in $U$. An *edge cover* of $G$ is a subset of the edges $F$ such that each vertex of $G$ has an edge in $F$ incident to it. Let

$$\begin{aligned} \alpha(G) &= \text{maximum size of an independent set of } G, \\ \rho(G) &= \text{minimum size of an edge cover of } G, \\ \tau(G) &= \text{minimum size of a vertex cover of } G, \\ \nu(G) &= \text{maximum size of a matching of } G. \end{aligned}$$

Prove that

$$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G).$$

*Hint:* For the first equality find a simple correspondence between independent sets of $G$ and vertex covers of $G$; for the second equality, prove first the inequality $|F| \leq |V| - \nu(G)$ for some edge cover $F$ of $G$ obtained from a maximum matching, then prove that $|M| \geq |V| - \rho(G)$ for some matching $M$ obtained from an minimum edge cover.

7. An *independent set* in a graph is a set of vertices with no edge between any pair of vertices in the set.

   (a) Give an integer program whose solutions are the independent sets in a graph $G = (V, E)$.

   (b) Now consider the polyhedron obtained by relaxing the integrality constraints. Show that if $G$ is bipartite then the vertices of this polyhedron are all integral.