1.8 SVD

Definition: $A \in \mathbb{C}^{m\times n}$, $m \geq n$, the SVD of A is

$$A = U\Sigma V^* \tag{1.26}$$

where, U, V unitary, $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

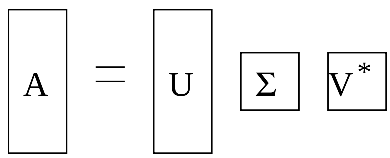

Figure 1.2: SVD.

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1.9 Existence and Uniqueness

Theorem: Every matrix A has an SVD. The singular values σ_i are uniquely determined and if A is square and the σ_i are distinct, then u_i and v_i are uniquely determined up to complex signs.

Proof: Let $\sigma_1 = \|A\|_2$. Let $\|v_1\|_2 = 1$ be such that $\|Av_1\|_2 = \|A\| = \sigma_1$. Let $u_1 = \frac{Av_1}{\sigma_1}$. Consider any extension of u_1 and v_1 to an orthonormal basis U_1 and V_1.

$$\begin{aligned} U_1 &= [u_1|\cdots] && (1.27)\\ V_1 &= [v_1|\cdots] && (1.28)\\ u_1^* A v_1 &= S \\ &= \begin{bmatrix} \sigma_1 & w^* \\ 0 & B \end{bmatrix} && (1.29) \end{aligned}$$

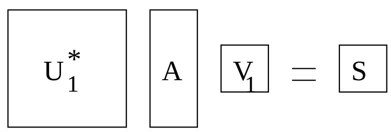

Figure 1.3: Proof.

Need to show $w = 0$. Assume $w \neq 0$, then

$$\begin{aligned} \left\| \begin{bmatrix} \sigma_1 & w^* \\ 0 & B \end{bmatrix} \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2 &\geq \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2^2 \\ &\geq \sqrt{\sigma_1^2 + \|w\|_2^2} \cdot \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2^2 \\ &> \sigma_1 \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2^2 . && (1.30) \end{aligned}$$

Therefore, $\sigma_1 = \|A\| = \|S\|_2 > \sigma_1 \rightarrow$ contradiction.
Proceed by induction:

$$\begin{aligned} B &= U_2 \Sigma V_2^* && (1.31)\\ A &= \underbrace{u_1 \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & V_2 \end{bmatrix}^* v_1^*}_{V^*} && (1.32) \end{aligned}$$

Chapter 2

2.1 Uniqueness of SVD (First Proof)

Assume

$$\sigma_1 = \|A\| = \|Av_1\| = \|Aw\|_2 \tag{2.1}$$

Need to show v_1 and w differ by a complex sign.
Let

$$v_2 = \frac{w - (v_1^* w) v_1}{\|w - (v_1^* w) v_1\|_2} \tag{2.2}$$

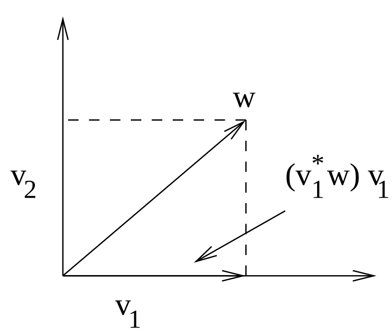

Figure 2.1: Uniqueness.

$$\|Av_2\|_2 \le \sigma_1 = \|A\|_2 \tag{2.3}$$

If

$$\|Av_2\| \le 1 \tag{2.4}$$

then

$$
\begin{aligned}
w &= v_1 c + v_2 s && (2.5)\\
|s|^2 + |c|^2 &= 1 && (2.6)\\
\|Aw\|_2 &= \|Av_1 c\| + \|Av_2 s\| < \sigma && (2.7)
\end{aligned}
$$

Same exercise as before.

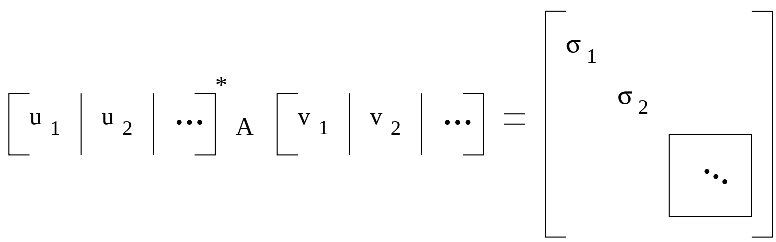

Figure 2.2: Exersise.

Chapter 3

3.1 Uniqueness of the SVD (Second Proof)

The singular values are uniquely determined and if A is square and $\sigma_1 > \cdots > \sigma_n \geq 0$ then the left and right singular values are uniquely determined up to complex signs.

Proof: $A = U\Sigma V^*$, A^*A-normal $\Rightarrow$ real eigenvalues. $A^*A = V\Sigma^2 V^* \Rightarrow \sigma_i^2$ eigenvalues of A^*A uniquely determined. σ_i-distinct $\Rightarrow \sigma_i^2$-distinct $\Rightarrow v_i$ uniquely determined (as the unique solutions to $A^*AV = \sigma_i^2 V$) up to a scalar but $\|v_i\|_2 = 1 \Rightarrow$ uniquely determined up to a complex sign.

3.2 Properties of the SVD

1. $r = \text{rank}(A)$ number of non-zero eigenvalues.
2. $\|A\|_2 = \sigma_1$, $\|A\|_F = \sqrt{\sum_{i=1}^n \sigma_i^2}$.
3. The non-zero σ_i's are the square roots of the non-zero eigenvalues of A^*A and AA^*.
4. $A = A^*$ (Hermitian) $\Rightarrow \sigma_i = |\lambda_i|$.
5. $|\det(A)| = \prod \sigma_i$

3.3 Best Rank k Approximation

For any $0 \leq \nu \leq r$, define

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*. \tag{3.1}$$

(called best rank ν approximation).

Proposition:

$$\|A - A_\nu\|_2 = \inf_{\text{rank } B \leq \nu} \|A - B\|_2 = \sigma_{\nu+1} \tag{3.2}$$

($\sigma_{n+1} \doteq 0$).
Proof: Suppose $\exists B$, rank $B \leq \nu$:

$$\|A - B\|_2 < \|A - A_\nu\| = \sigma_{\nu+1} \tag{3.3}$$

Therefore, $\exists (n-\nu)$-dimensional subspace $W \subseteq \mathbb{C}^n : \forall w \in W$ we have $Bw = 0 \Rightarrow Aw = (A-B)w$

$$\|Aw\|_2 = \|(A - B)w\|_2 \leq \|A - B\|_2 \cdot \|w\|_2 < \sigma_{\nu+1} \|w\|_2 \tag{3.4}$$

Therefore, we have a $(n - \nu)$-dimensional subspace W:

$$\|Aw\| < \sigma_{\nu+1} \|w\| \tag{3.5}$$

Let $\overline{w} = \text{span}(v_1, \cdots, v_{\nu+1})$, then $\dim(\overline{w}) = \nu + 1$ and $\|A\overline{w}\|_2 \geq \sigma_{\nu+1} \|\overline{w}\|_2$, for any $\overline{w} \in \overline{W}$.
$\dim(W) + \dim(\overline{W}) = n + 1 \Rightarrow W \cap \overline{W} \neq \{0\}$, therefore, $w \in W \cap \overline{W}$; contradiction.