Chapter 13

13.1 LR Iteration

Chrolesky Factorization:

$$B_i^T B_i = B_{i-1} B_{i-1}^T \tag{13.1}$$

$$\begin{bmatrix} a_1' & & & \\ b_1' & \ddots & & \\ & \ddots & \ddots & \\ & & b_n' & a_n' \end{bmatrix} \times \begin{bmatrix} a_1' & b_1' & & \\ & \ddots & \ddots & \\ & & \ddots & b_n' \\ & & & a_n' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & & \\ & \ddots & \ddots & \\ & & \ddots & b_n \\ & & & a_n \end{bmatrix} \times \begin{bmatrix} a_1 & & & \\ b_1 & \ddots & & \\ & \ddots & \ddots & \\ & & b_n & a_n \end{bmatrix} \tag{13.2}$$

$$\begin{bmatrix} (a_1')^2 & a_1'b_1' & & \\ a_1'b_1' & (b_1')^2 + (a_1')^2 & a_2'b_2' & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} a_1^2 + b_1^2 & b_1 a_2 & & \\ b_1 a_2 & a_2^2 + b_2^2 & b_2 a_3 & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix} \tag{13.3}$$

Therefore:

$$(a_1')^2 = a_1^2 + b_1^2 \tag{13.4}$$

$$a_1'b_1' = b_1 a_2 \tag{13.5}$$

$$\vdots \tag{13.6}$$

$$a_i'b_i' = b_i a_{i+1} \tag{13.7}$$

$$(b_{i-1}')^2 + (a_i')^2 = a_i^2 + b_i^2 \tag{13.8}$$

$$(a_i')^2 = a_i^2 + b_i^2 - (b_{i-1}')^2 \tag{13.9}$$

Define:

$$
\begin{aligned}
q_i &= a_i^2 && (13.10)\\
e_i &= b_i^2 && (13.11)
\end{aligned}
$$

then

$$
\begin{aligned}
q_i^{'} &= q_i + e_i - e_{i-1}^{'} && (13.12)\\
e_i^{'} &= e_i \frac{q_{i+1}}{q_i^{'}} && (13.13)\\
d_i &= q_i - e_{i-1}^{'} \\
&= q_i\left(\frac{q_{i-1}^{'} - e_{i-1}}{q_{i-1}^{'}}\right) \\
&= \frac{q_i}{q_{i-1}^{'}} d_{i-1} && (13.14)\\
q_i^{'} &= d_i + e_i && (13.15)\\
e_i^{'} &= e_i \frac{q_{i+1}}{q_i^{'}} && (13.16)\\
d_{i+1} &= d_i \frac{q_{i+1}}{q_i^{'}} && (13.17)
\end{aligned}
$$

Shifts can also be incorporated. The algorithm is stable and highly accurate. The stability comes from the fact that small relative perturbations in the entries of the bidiagonal Cholesky factor of a tridiagonal matrix only cause small relative perturbations in the eigenvalues of the tridiagonal matrix. All errors propagate multiplicatively (and this is where the accuracy comes from) because we only add, multiply and divide positive quantities.