# 3 Homework Solutions

## 18.335 - Fall 2004

### 3.1 Trefethen 10.1

**(a)** $H = I - 2vv^*$ where $\|v\| = 1$. If $v^*u = 0$ ( $u$ is perpendicular to $v$ ), then $Hu = u - 2vv^*u = u$. So 1 is an eigenvalue with multiplicity $n-1$ ( there are $n-1$ linearly independent eigenvectors perpendicular to $v$ ). Also $Hv = v - 2vv^*v = v - 2v = -v$, so $-1$ is an eigenvalue of $H$. The geometric interpretation is given by the fact that reflection of $v$ is $-v$, and reflection of any vector perpendicular to $v$ is $v$ itself.

**(b)** $\det H = \prod_{i=1}^{n} \lambda_i = (-1)\,1^{n-1} = -1.$

**(c)** $H^*H = (I - 2vv^*)^* (I - 2vv^*) = I - 4vv^* + 4vv^*vv^* = I$. So the singular values are all 1's.

### 3.2 Let $B$ be an $n \times n$ upper bidiagonal matrix. Describe an algorithm for computing the condition number of $B$ measured in the infinity norm in $\mathcal{O}(n)$ time.

The condition number of $B$ in the infinity norm is defined as:

$$\kappa_\infty(B) = \left\|B^{-1}\right\|_\infty \|B\|_\infty$$

We have to compute these two matrix norms separately. For $\|B\|_\infty$, the operation count is $\mathcal{O}(n)$ since only $n-1$ operations (corresponding to row sums for the first $n-1$ rows) are required. In order to calculate $\left\|B^{-1}\right\|_\infty$ we need to compute $B^{-1}$ first. To do so, let $C = B^{-1}$. Performing the matrix multiplication $BC = I$, one can see that the inverse is an upper triangular matrix whose entries are given by:

$$c_{i,j} = \begin{cases} 0 & ,i > j \\ \dfrac{1}{b_{i,i}} & ,i = j \\ \dfrac{1}{b_{i,i}} \displaystyle\prod_{k=i}^{j-1} \left(-\frac{b_{k,k+1}}{b_{k+1,k+1}}\right) & ,i < j \end{cases}$$

This enables us to compute the infinity norm of $B^{-1}$ :

$$\left\|B^{-1}\right\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |c_{i,j}| = \max_i \underbrace{\frac{1}{|b_{i,i}|} \left(1 + \sum_{j=i+1}^{n} \left|\prod_{k=i}^{j-1} \left(-\frac{b_{k,k+1}}{b_{k+1,k+1}}\right)\right|\right)}_{P_i}$$

Normally, doing this directly would require $\mathcal{O}\left(n^2\right)$ flops. We can avoid this many flops by making some simplifications. Let:

$$d_k = \left| \frac{b_{k,k+1}}{b_{k+1,k+1}} \right|$$

and notice that the row sum, $P_i$ can be written as:

$$\begin{aligned} P_i &= \frac{1}{|b_{i,i}|} \left( 1 + \sum_{j=i+1}^{n} \prod_{k=i}^{j-1} d_k \right) = \frac{1}{|b_{i,i}|} \left( 1 + d_i + \sum_{j=i+2}^{n} \prod_{k=i}^{j-1} d_k \right) \\ &= \frac{1}{|b_{i,i}|} \left[ 1 + d_i \left( 1 + \sum_{j=i+2}^{n} \prod_{k=i+1}^{j-1} d_k \right) \right] = \frac{1}{|b_{i,i}|} \left( 1 + d_i \left| b_{i+1,i+1} \right| P_{i+1} \right) \\ &= \frac{1}{|b_{i,i}|} \left( 1 + \left| b_{i,i+1} \right| P_{i+1} \right) \end{aligned}$$

Thus knowing $P_{i+1}$ we can calculate $P_i$ in 5 operations. Hence we have to start from the $n$-th row and proceed backwards. So our algorithm to compute $\left\| B^{-1} \right\|_\infty$ is:

$$P_n = \frac{1}{|b_{n,n}|} = \left\| B^{-1} \right\|_\infty \qquad \Big\} \text{ 2 flops}$$

**for** i=n-1 **to** 1

$$P_i = \frac{1 + \left| b_{i,i+1} \right| P_{i+1}}{|b_{i,i}|}$$

$$\left\| B^{-1} \right\|_\infty = \max\left(P_i, P_{i+1}\right)$$

**end**

$\Big\}$ $\mathcal{O}(n)$ flops

Since both $\left\| B^{-1} \right\|_\infty$ and $\| B \|_\infty$ require $\mathcal{O}(n)$ flops, so does $\kappa_\infty(B)$.