18.330 :: Homework 6 :: Spring 2012 :: Due Thursday April 26

1. (7pts) Consider the mixed Neumann-Dirichlet boundary-value problem

$$-u'' = f, \qquad x \in [0,1], \qquad u'(0) = 0, \qquad u(1) = 0.$$

   a) Find the eigenvalues and eigenfunctions for this problem, i.e., find all the possible λ_n and $v_n(x)$ such that

$$-v_n'' = \lambda_n v_n, \qquad v_n'(0) = 0, \qquad v_n(1) = 0.$$

   Draw (by hand) the first three eigenfunctions. [Hint: the v_j are sines or cosines. The boundary conditions determine which ones to consider.]

   b) Consider the basic finite-difference discretization of the second derivative on the grid $x_j = jh$ with $h = 1/N$, and $j = 0, \ldots, N-1$. For simplicity discretize the Neumann condition as

$$\frac{U_1 - U_0}{h} = 0.$$

   Write down the matrix that arises in the discretized system, and show code that creates it in Matlab. We'll call this matrix T.

   c) Use Matlab's `eig` command to find the eigenvalues and eigenvectors of the matrix T for some reasonably large value of N (larger than 100). Plot the first three eigenvectors. Do they match your theoretical prediction in the continuous case? [Hint: the eigenvalues obtained from eig may not be sorted in increasing order. Make sure you sort the eigenvalues and eigenvectors the same way.]

   d) Use $f(x) = 1$ as right-hand side, and discretize it as $F_j = 1$. Solve for U in $TU = F$ (using Matlab's backslash) for various values of N, and document the order of accuracy by plotting the error in a log-log plot. To measure the error $e = u - U$, use the quadratic norm

$$\|u - U\| = \sqrt{h \sum_{j=0}^{N-1} (u(x_j) - U_j)^2}, \qquad \text{(notice the presence of } h.\text{)}$$

   [Hint: As reference solution, use the U that you obtain form a very fine grid. In order to compare solutions defined on different grids, make sure that those grids are nested.]

   e) The local truncation error, as measured in the quadratic norm above, is $O(h)$ because the boundary condition is only first-order accurate. What property of T is needed to justify that the overall error is also $O(h)$ in the same norm?

2. (2pts) Consider the Neumann problem

$$-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f, \qquad x \in [0,1], \qquad u'(0) = u'(1) = 0.$$

   a) Prove that if $f = 1$, then this equation cannot have a solution. [Hint: integrate both sides of the equation.]

   b) Find a condition on f which guarantees that the equation has a solution.

3. (1pt) Consider the matrix

$$A = \begin{pmatrix} 2 & a \\ 1 & 0 \end{pmatrix}$$

   a) For what values of a is the matrix invertible?

   b) Without computing the eigenvalues and eigenvectors, find the value of a for which A has an orthonormal basis of eigenvectors. Explain your answer.