Exercises 4

(1) [2–] Let M be a matroid on a linearly ordered set. Show that $\mathrm{BC}(M) = \mathrm{BC}(\widehat{M})$, where $\widehat{M}$ is defined by equation (23).

(2) [2+] Let M be a matroid of rank at least one. Show that the coefficients of the polynomial $\chi_M(t)/(t-1)$ alternate in sign.

(3) (a) [2+] Let L be finite lattice for which every element has a unique complement. Show that L is isomorphic to a boolean algebra B_n.

(b) [3] A lattice L is *distributive* if

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \end{aligned}$$

for all $x, y, z \in L$. Let L be an infinite lattice with $\hat{0}$ and $\hat{1}$. If every element of L has a unique complement, then is L a distributive lattice?

(4) [3–] Let x be an element of a geometric lattice L. Show that the following four conditions are equivalent.

(i) x is a modular element of L.

(ii) If $x \wedge y = \hat{0}$, then

$$\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \vee y).$$

(iii) If x and y are complements, then $\mathrm{rk}(x) + \mathrm{rk}(y) = n$.

(iv) All complements of x are incomparable.

(5) [2+] Let x, y be modular elements of a geometric lattice L. Show that $x \wedge y$ is also modular.

(6) [2] Let L be a geometric lattice. Prove or disprove: if x is modular in L and y is modular in the interval $[x, \hat{1}]$, then y is modular in L.

(7) [2–] Let L and L' be finite lattices. Show that if both L and L' are geometric (respectively, atomic, semimodular, modular) lattices, then so is $L \times L'$.

(8) [2] Let G be a (loopless) connected graph and $v \in V(G)$. Let $A = V(G) - v$ and $\pi = \{A, v\} \in L_G$. Suppose that whenever $av, bv \in E(G)$ we have $ab \in E(G)$. Show that π is a modular element of L_G.

(9) [2+] Generalize the previous exercise as follows. Let G be a doubly-connected graph with lattice of contractions L_G. Let $\pi \in L_G$. Show that the following two conditions are equivalent.

(a) π is a modular element of L_G.

(b) π satisfies the following two properties:

(i) At most one block B of π contains more than one vertex of G.

(ii) Let H be the subgraph induced by the block B of (i). Let K be any connected component of the subgraph induced by $G - B$, and let H_1 be the graph induced by the set of vertices in H that are connected to some vertex in K. Then H_1 is a clique (complete subgraph) of G.

(10) [2+] Let L be a geometric lattice of rank n, and fix $x \in L$. Show that

$$\chi_L(t) = \sum_{\substack{y \in L \\ x \wedge y = \hat{0}}} \mu(y) \chi_{L_y}(t) t^{n - \mathrm{rk}(x \vee y)},$$

where L_y is the image of the interval $[\hat{0}, x]$ under the map $z \mapsto z \vee y$.

(11) [2+] Let $\mathcal{I}(M)$ be the set of independent sets of a matroid M. Find another matroid N and a labeling of its points for which $\mathcal{I}(M) = \mathrm{BC}_r(N)$, the reduced broken circuit complex of N.

(12) (a) [2+] If Δ and Γ are simplicial complexes on disjoint sets A and B, respectively, then define the *join* $\Delta * \Gamma$ to be the simplicial complex on the set $A \cup B$ with faces $F \cup G$, where $F \in \Delta$ and $G \in \Gamma$. (E.g., if Γ consists of a single point then $\Delta * \Gamma$ is the *cone* over Δ. If Γ consists of two disjoint points, then $\Delta * \Gamma$ is the *suspension* of Δ.) We say that Δ and Γ are *join-factors* of $\Delta * \Gamma$. Now let M be a matroid and $S \subset M$ a modular flat, i.e., S is a modular element of L_M. Order the points of M such that if $p \in S$ and $q \notin S$, then $p < q$. Show that $\mathrm{BC}(S)$ is a join-factor of $\mathrm{BC}(M)$. Deduce that $\chi_M(t)$ is divisible by $\chi_S(t)$.

(b) [2+] Conversely, let M be a matroid and $S \subset M$. Label the points of M so that if $p \in S$ and $q \notin S$, then $p < q$. Suppose that $\mathrm{BC}(S)$ is a join-factor of $\mathrm{BC}(M)$. Show that S is modular.

(13) [2] Do Exercise 7, this time using Theorem 4.12 (the Broken Circuit Theorem).

(14) [1] Show that all geometric lattices of rank two are supersolvable.

(15) [2] Give an example of two nonisomorphic supersolvable geometric lattices of rank 3 with the same characteristic polynomials.

(16) [2] Prove Proposition 4.11: if G is a graph with blocks $G_1, \ldots, G_k$, then $L_G \cong L_{G_1} \times \cdots \times L_{G_k}$.

(17) [2+] Give an example of a nonsupersolvable geometric lattice of rank three whose characteristic polynomial has only integer zeros.

(18) [2] Let L_1 and L_2 be geometric lattices. Show that L_1 and L_2 are supersolvable if and only if $L_1 \times L_2$ is supersolvable.

(19) [3–] Let L be a supersolvable geometric lattice. Show that every interval of L is also supersolvable.

(20) [2] (a) Find the number of maximal chains of the partition lattice Π_n.

(b) Find the number of modular maximal chains of Π_n.

(21) Let M be a matroid with a linear ordering of its points. The *internal activity* of a basis B is the number of points $p \in B$ such that $p < q$ for all points $q \neq p$ not in the closure $\overline{B - p}$ of $B - p$. The *external activity* of B is the number of points $p' \in M - B$ such that $p' < q'$ for all $q' \neq p'$ contained in the unique circuit that is a subset of $B \cup \{p'\}$. Define the *Crapo beta invariant* of M by

$$\beta(M) = (-1)^{\mathrm{rk}(M)-1} \chi'_M(1),$$

where $'$ denotes differentiation.

(a) [1+] Show that $1 - \chi'_M(1) = \psi(\mathrm{BC}_r)$, the Euler characteristic of the reduced broken circuit complex of M.

(b) [3–] Show that $\beta(M)$ is equal to the number of bases of M with internal activity 0 and external activity 0.

(c) [2] Let $\mathcal{A}$ be a real central arrangement with associated matroid $M_{\mathcal{A}}$. Suppose that $\mathcal{A} = c\mathcal{A}'$ for some arrangement $\mathcal{A}'$, where $c\mathcal{A}'$ denotes the cone over $\mathcal{A}'$. Show that $\beta(M_{\mathcal{A}}) = b(\mathcal{A}')$.

(d) [2+] With $\mathcal{A}$ as in (c), let H' be a (proper) translate of some hyperplane $H \in \mathcal{A}$. Show that $\beta(M_{\mathcal{A}}) = b(\mathcal{A} \cup \{H'\})$.

Exercises 5

(1) [2] Verify equation (37), viz.,

$$\chi_{\mathcal{A}(D_n)}(t) = (t-1)(t-3)\cdots(t-(2n-3)\cdot(t-n+1).$$

(2) [2] Draw a picture of the projectivization of the Coxeter arrangement $\mathcal{A}(B_3)$, similar to Figure 1 of Lecture 1.

(3) (a) [2] An *embroidered permutation* of $[n]$ consists of a permutation w of $[n]$ together with a collection $\mathcal{E}$ of ordered pairs (i,j) such that:
- $1 \le i < j \le n$ for all $(i,j) \in \mathcal{E}$.
- If (i,j) and (h,k) are distinct elements of $\mathcal{E}$, then it is false that $i \le h \le k \le j$.
- If $(i,j) \in \mathcal{E}$ then $w(i) < w(j)$.

For instance, the three embroidered permutations $(w,\mathcal{E})$ of $[2]$ are given by $(12,\emptyset)$, $(12,\{(1,2)\})$, and $(21,\emptyset)$. Give a bijective proof that the number $r(\mathcal{S}_n)$ of regions of the Shi arrangement $\mathcal{S}_n$ is equal to the number of embroidered permutations of $[n]$.

(b) [2+] A *parking function* of length n is a sequence $(a_1,\ldots,a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \le b_2 \le \cdots \le b_n$ satisfies $b_i \le i$. For instance, the parking functions of length three are 11, 12, 21. Give a bijective proof that the number of parking functions of length n is equal to the number of embroidered permutations of $[n]$.

(c) [3–] Give a combinatorial proof that the number of parking functions of length n is equal to $(n+1)^{n-1}$.

(4) [2+] Show that if $\mathcal{S}_n$ denotes the Shi arrangement, then the cone $c\mathcal{S}_n$ is not supersolvable for $n \ge 3$.

(5) [2] Show that if $f:\mathbb{P}\to R$ and $h:\mathbb{N}\to R$ are related by equation (40) (with $h(0)=1$), then equation (39) holds.

(6) (a) [2] Compute the characteristic polynomial of the arrangement $\mathcal{B}'_n$ in $\mathbb{R}^n$ with defining polynomial

$$Q(x) = (x_1 - x_n - 1)\prod_{1\le i<j\le n}(x_i - x_j).$$

In other words, $\mathcal{B}'_n$ consists of the braid arrangement together with the hyperplane $x_1 - x_n = 1$.

(b) [5–] Is $c\mathcal{B}'_n$ (the cone over $\mathcal{B}'_n$) supersolvable?

(7) [2+] Let $1 \le k \le n$. Find the characteristic polynomial of the arrangement $\mathcal{S}_{n,k}$ in $\mathbb{R}^n$ defined by

$$\begin{array}{rl} x_i - x_j = 0 & \text{for} \quad 1 \le i < j \le n \\ x_i - x_j = 1 & \text{for} \quad 1 \le i < j \le k. \end{array}$$

(8) [2+] Let $1 \le k \le n$. Find the characteristic polynomial of the arrangement $\mathcal{C}_{n,k}$ in $\mathbb{R}^n$ defined by

$$\begin{array}{rl} x_i = 0 & \text{for} \quad 1 \le i \le n \\ x_i \pm x_j = 0 & \text{for} \quad 1 \le i < j \le n \\ x_i + x_j = 1 & \text{for} \quad 1 \le i < j \le k. \end{array} .$$

In particular, show that $r(\mathcal{C}_{n,k}) = 2^{n-k} n! \binom{2k}{k}$.

(9) (a) [2+] Let $\mathcal{A}_n$ be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = 0$ for all i, $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i \neq j$. Show that

$$\chi_{\mathcal{A}_n}(t) = (t-1)(t-n-2)_{n-1},$$

where $(x)_m = x(x-1)\cdots(x-m+1)$. In particular, $r(\mathcal{A}_n) = 2(2n+1)!/(n+2)!$. Can this be seen combinatorially? (This last question has not been worked on.)

(b) [2+] Now let $\mathcal{A}_n$ be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = x_j$ for all $i < j$ and $x_i = 2x_j$ for all $i \neq j$. Show that

$$\chi_{\mathcal{A}_n}(t) = (t-1)(t-n-2)_{n-3}(t^2 - (3n-1)t + 3n(n-1)).$$

In particular, $r(\mathcal{A}_n) = 6n^2(2n-1)!/(n+2)!$. Again, a combinatorial proof can be asked for.

(c) [5–] Modify. For instance, what about the arrangement with hyperplanes $x_i = 0$ for all i, $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i < j$? Or $x_i = 0$ for all i, $x_i = x_j$ for all $i < j$, $x_i = 2x_j$ for all $i \neq j$, and $x_i = 3x_j$ for all $i \neq j$?

(10) (a) [2+] For $n \geq 1$ let $\mathcal{A}_n$ be an arrangement in $\mathbb{R}^n$ such that every $H \in \mathcal{A}_n$ is parallel to a hyperplane of the form $x_i = cx_j$, where $c \in \mathbb{R}$. Just as in the definition of an exponential sequence of arrangements, define for every subset S of $[n]$ the arrangement

$$\mathcal{A}_n^S = \{H \in \mathcal{A}_n \,:\, H \text{ is parallel to some } x_i = cx_j, \text{ where } i,j \in S\}.$$

Suppose that for every such S we have $L_{\mathcal{A}_n^S} \cong L_{\mathcal{A}_k}$, where $k = \#S$. Let

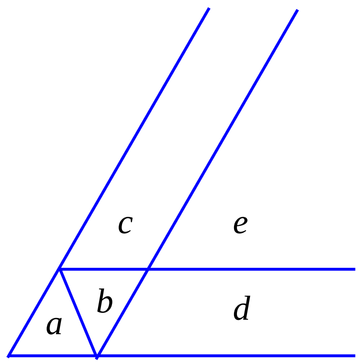
$$\begin{aligned} F(x) &= \sum_{n\geq 0} (-1)^n r(\mathcal{A}_n)\frac{x^n}{n!} \\ G(x) &= \sum_{n\geq 0} (-1)^{\mathrm{rk}(\mathcal{A}_n)} b(\mathcal{A}_n)\frac{x^n}{n!}. \end{aligned}$$

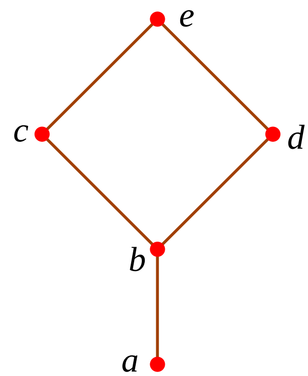
Show that

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(t)\frac{x^n}{n!} = \frac{G(x)^{(t+1)/2}}{F(x)^{(t-1)/2}}. \tag{48}$$

(b) [2] Simplify equation (48) when each $\mathcal{A}_n$ is a central arrangement. Make sure that your simplification is valid for the braid arrangement and the coordinate hyperplane arrangement.

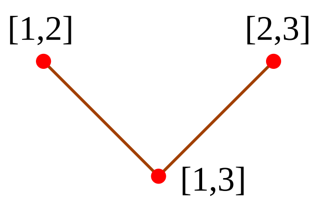
(11) [2+] Let $\mathcal{R}_0(\mathcal{C}_n)$ denote the set of regions of the Catalan arrangement $\mathcal{C}_n$ contained in the regions $x_1 > x_2 > \cdots > x_n$ of $\mathcal{B}_n$. Let $\hat{R}$ be the unique region in $\mathcal{R}_0(\mathcal{C}_n)$ whose closure contains the origin. For $R \in \mathcal{R}_0(\mathcal{C}_n)$, let X_R be the set of hyperplanes $H \in \mathcal{C}_n$ such that $\hat{R}$ and R lie on different sides of H. Let $W_n = \{X_R \,:\, R \in \mathcal{R}_0(\mathcal{C}_n)\}$, ordered by inclusion.



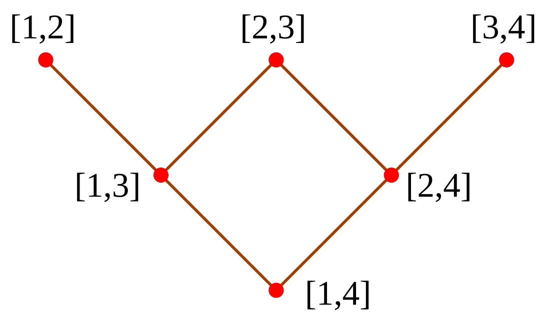


W_3

Let P_n be the poset of intervals $[i, j]$, $1 \leq i < j \leq n$, ordered by *reverse* inclusion.



P_3



P_4

Show that $W_n \cong J(P_n)$, the lattice of order ideals of P_n. (An *order ideal* of a poset P is a subset $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. Define $J(P)$ to be the set of order ideals of P, ordered by inclusion. See [**18**, Thm. 3.4.1].)

(12) [2] Use the finite field method to prove that

$$\chi_{\mathcal{C}_n}(t) = t(t-n-1)(t-n-2)(t-n-3)\cdots(t-2n+1),$$

where $\mathcal{C}_n$ denotes the Catalan arrangement.

(13) [2+] Let $k \in \mathbb{P}$. Find the number of regions and characteristic polynomial of the *extended Catalan arrangement*

$$\mathcal{C}_n(k): \ x_i - x_j = 0, \pm 1, \pm 2, \dots, \pm k, \ \text{for } 1 \leq i < j \leq n.$$

Generalize Exercise 11 to the arrangements $\mathcal{C}_n(k)$.

(14) [3–] Let $\mathcal{S}_n^B$ denote the arrangement

$$\begin{aligned} x_i \pm x_j &= 0, 1, \qquad 1 \leq i < j \leq n \\ 2x_i &= 0, 1, \qquad 1 \leq i \leq n, \end{aligned}$$

called the *Shi arrangement of type B*. Find the characteristic polynomial and number of regions of $\mathcal{S}_n^B$. Is there a "nice" bijective proof of the formula for the number of regions?

(15) [5–] Let $1 \leq k \leq n$. Find the number of regions (or more generally the characteristic polynomial) of the arrangement (in $\mathbb{R}^n$)

$$x_i - x_j = \begin{cases} 1, & 1 \leq i \leq k \\ 2, & k+1 \leq i \leq n, \end{cases}$$

for all $i \neq j$. Thus we are counting interval orders on $[n]$ where the elements $1, 2, \dots, k$ correspond to intervals of length one, while $k+1, \dots, n$ correspond to intervals of length two. Is it possible to count such interval orders up to isomorphism (i.e., the unlabelled case)? What if the length 2 is replaced instead by a generic length a?

(16) [2+] A *double semiorder* on $[n]$ consists of two binary relations $<$ and $\ll$ on $[n]$ that arise from a set $x_1, \dots, x_n$ of real numbers as follows:

$$\begin{array}{lll} i < j & \text{if} & x_i < x_j - 1 \\ i \ll j & \text{if} & x_i < x_j - 2. \end{array}$$

If we associate the interval $I_i = [x_i - 2, x_i]$ with the point x_i, then we are specifying whether I_i lies to the left of the midpoint of I_j, entirely to the left of I_j, or neither. It should be clear what is meant for two double semiorders to be *isomorphic*.

(a) [2] Draw interval diagrams of the 12 nonisomorphic double semiorders on $\{1, 2, 3\}$.

(b) [2] Let $\rho_2(n)$ denote the number of double semiorders on $[n]$. Find an arrangement $\mathcal{I}_n^{(2)}$ satisfying $r(\mathcal{I}_n^{(2)}) = \rho_2(n)$.

(c) [2+] Show that the number of nonisomorphic double semiorders on $[n]$ is given by $\frac{1}{2n+1}\binom{3n}{n}$.

(d) [2–] Let $F(x) = \sum_{n \geq 0} \frac{1}{2n+1}\binom{3n}{n} x^n$. Show that

$$\sum_{n \geq 0} \rho_2(n) \frac{x^n}{n!} = F(1 - e^{-x}).$$

(e) [2] Generalize to "k-semiorders," where ordinary semiorders (or unit interval orders) correspond to $k = 1$ and double semiorders to $k = 2$.

(17) [1+] Show that intervals of lengths $1, 1.0001, 1.001, 1.01, 1.1$ cannot form an interval order isomorphic to $\mathbf{4} + \mathbf{1}$, but that such an interval order *can* be formed if the lengths are $1, 10, 100, 1000, 10000$.

(18) [5–] What more can be said about interval orders with generic interval lengths? For instance, consider the two cases: (a) interval lengths very near each other (e.g., 1, 1.001, 1.01, 1.1), and (b) interval lengths superincreasing (e.g., 1, 10, 100, 1000). Are there finitely many obstructions to being such an interval order? Can the number of unlabelled interval orders of each type be determined? (Perhaps the numbers are the same, but this seems unlikely.)

(19) (a) [3] Let $\mathcal{L}_n$ denote the Linial arrangement, say in $\mathbb{R}^n$. Show that

$$\chi_{\mathcal{L}_n}(t) = \frac{t}{2^n} \sum_{k=1}^{n} \binom{n}{k} (t-k)^{n-1}.$$

(b) [1+] Deduce from (a) that

$$\frac{\chi_{\mathcal{L}_n}(t)}{t} = \frac{(-1)^n \chi_{\mathcal{L}_n}(-t+n)}{-t+n}.$$

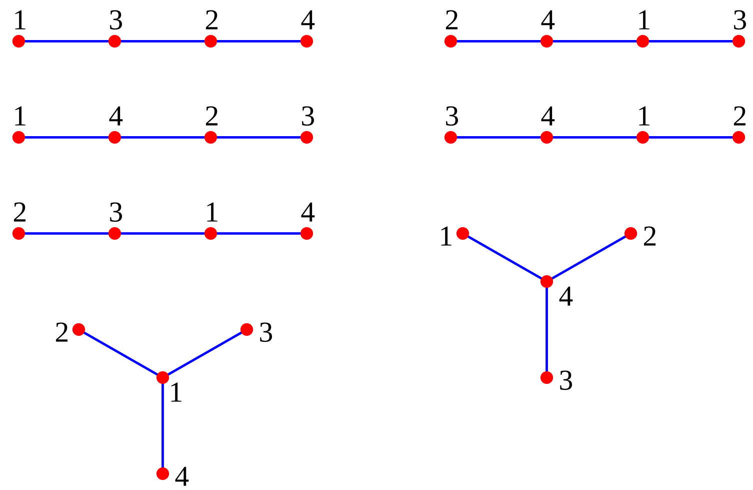


Figure 10. The seven alternating trees on the vertex set [4]

(20) (a) [3–] An *alternating tree* on the vertex set $[n]$ is a tree on $[n]$ such that every vertex is either less than all its neighbors or greater than all its neighbors. Figure 10 shows the seven alternating trees on [4]. Deduce from Exercise 19(a) that $r(\mathcal{L}_n)$ is equal to the number of alternating trees on $[n+1]$.

(b) [5] Find a bijective proof of (a), i.e., give an explicit bijection between the regions of $\mathcal{L}_n$ and the alternating trees on $[n+1]$.

(21) [3–] Let
$$\chi_{\mathcal{L}_n}(t) = a_n t^n - a_{n-1}t^{n-1} + \cdots + (-1)^{n-1}a_1 t.$$
Deduce from Exercise 19(a) that a_i is the number of alternating trees on the vertex set $0, 1, \dots, n$ such that vertex 0 has degree (number of adjacent vertices) i.

(22) (a) [2+] Let $P(t) \in \mathbb{C}[t]$ have the property that every (complex) zero of $P(t)$ has real part a. Let $z \in \mathbb{C}$ satisfy $|z| = 1$. Show that every zero of the polynomial $P(t-1) + zP(t)$ has real part $a + \frac{1}{2}$.

(b) [2+] Deduce from (a) and Exercise 19(a) that every zero of the polynomial $\chi_{\mathcal{L}_n}(t)/t$ has real part $n/2$. This result is known as the "Riemann hypothesis for the Linial arrangement."

(23) (a) [2–] Compute $\lim_{n\to\infty} b(\mathcal{S}_n)/r(\mathcal{S}_n)$, where $\mathcal{S}_n$ denotes the Shi arrangement.

(b) [3] Do the same for the Linial arrangement $\mathcal{L}_n$.

(24) [2+] Let $\mathcal{L}_n$ denote the Linial arrangement in $\mathbb{R}^n$. Fix an integer $r \neq 0, \pm 1$, and let $\mathcal{M}_n(r)$ be the arrangement in $\mathbb{R}^n$ defined by $x_i = rx_j$, $1 \le i < j \le n$, together with the coordinate hyperplanes $x_i = 0$. Find a relationship between $\chi_{\mathcal{L}_n}(t)$ and $\chi_{\mathcal{M}_n(r)}(t)$ *without* explicitly computing these characteristic polynomials.

(25) (a) [3–] A *threshold graph* on $[n]$ may be defined recursively as follows: (i) the empty graph $\emptyset$ is a threshold graph, (ii) if G is a threshold graph, then so is the disjoint union of G and a single vertex, and (iii) if G is a threshold graph, then so is the graph obtained by adding a new vertex v and connecting it to every vertex of G. Let $\mathcal{T}_n$ denote the threshold arrangement. Show that $r(\mathcal{T}_n)$ is the number of threshold graphs on $[n]$.

(b) [2+] Deduce from (a) that

$$\sum_{n\geq 0} r(\mathcal{T}_n)\frac{x^n}{n!} = \frac{e^x(1-x)}{2-e^x}.$$

(c) [1+] Deduce from Exercise 10 that

$$\sum_{n\geq 0} \chi_{\mathcal{T}_n}(t)\frac{x^n}{n!} = (1+x)(2e^x-1)^{(t-1)/2}.$$

(26) [5–] Let

$$\chi_{\mathcal{T}_n}(t) = t^n - a_{n-1}t^{n-1} + \cdots + (-1)^n a_0.$$

For instance,

$$\begin{aligned} \chi_{\mathcal{T}_3}(t) &= t^3 - 3t^2 + 3t - 1 \\ \chi_{\mathcal{T}_4}(t) &= t^4 - 6t^3 + 15t^2 - 17t + 7 \\ \chi_{\mathcal{T}_5}(t) &= t^5 - 10t^4 + 45t^3 - 105t^2 + 120t - 51. \end{aligned}$$

By Exercise 25(a), $a_0 + a_1 + \cdots + a_{n-1} + 1$ is the number of threshold graphs on the vertex set $[n]$. Give a combinatorial interpretation of the numbers a_i as the number of threshold graphs with a certain property.

(27) (a) [1+] Find the number of regions of the "Linial threshold arrangement"

$$x_i + x_j = 1, \quad 1 \leq i < j \leq n.$$

(b) [5–] Find the number of regions, or even the characteristic polynomial, of the "Shi threshold arrangement"

$$x_i + x_j = 0, 1, \quad 1 \leq i < j \leq n.$$

(28) [3–] Let $\mathcal{A}_n$ denote the "generic threshold arrangement" (in $\mathbb{R}^n$) $x_i + x_j = a_{ij}$, $1 \leq i < j \leq n$, where the a_{ij}'s are generic. Let

$$T(x) = \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!},$$

the generating function for labelled trees on n vertices. Let

$$R(x) = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!},$$

the generating function for *rooted* labelled trees on n vertices. Show that

$$\begin{aligned} \sum_{n\geq 0} r(\mathcal{A}_n)\frac{x^n}{n!} &= e^{T(x)-\frac{1}{2}R(x)}\left(\frac{1+R(x)}{1-R(x)}\right)^{1/4} \\ &= 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 54\frac{x^4}{4!} + 533\frac{x^5}{5!} + 6934\frac{x^6}{6!} + \cdots. \end{aligned}$$

(29) [2+] Fix $k, n \geq 1$ and $r \geq 0$. Let $f(k,n,r)$ be the number of $k\times n$ $(0,1)$-matrices A over the rationals such that all rows of A are distinct, every row has at least one 1, and $\mathrm{rank}(A) = r$. Let $g_n(q)$ be the number of n-tuples $(a_1,\ldots,a_n) \in \mathbb{F}_q^n$ such that no nonempty subset of the entries sums to 0 (in $\mathbb{F}_q$). Show that for $p \gg 0$, where $q = p^d$, we have

$$g_n(q) = \sum_{k,r} \frac{(-1)^k}{k!} f(k,n,r) q^{n-r}.$$

(The case $k = 0$ is included, corresponding to the empty matrix, which has rank 0.)