Lecture Thirteen: Method of Stationary Phase

In 1911, E. Rutherford performed an experiment in which α-particles were made to pass through a thin metal foil. He measured the α-particles emerging from the foil at an angle, and found that the rate for such events, while being very low, is much higher than what had been expected. More precise experiments performed later showed that, with the scattering angle at 90 degrees, the rate of such events is 1 event out of 8000 events. While such an event is rare, it is at least by a factor 10^{10} more frequent than what had been predicted by J. J. Thomson's atomic model. Rutherford concluded that this means that an atom has a nucleus which is much, much smaller than even the tiny atom. His discovery ushered in the nuclear age. Scientists have since found that if we imagine an atom to be of the size of the Earth we live on, then in proportion the nucleus would be not much bigger than a campus gymnasium. Rutherford's experiment illustrates that it is not enough to know that an event is rare. We must also know how rare it is.

While the scattering of α-particles from the atoms in the metal foil obeys the laws of quantum mechanics, the phenomenon Rutherford observed falls within a general pattern of wave interference. Consider, for example, a monochromatic wave with the amplitude $e^{i\vec{k}\cdot\vec{x}}$ passing through two slits in a wall, arriving at a point on a screen behind the wall. We illustrate this in the figure below.
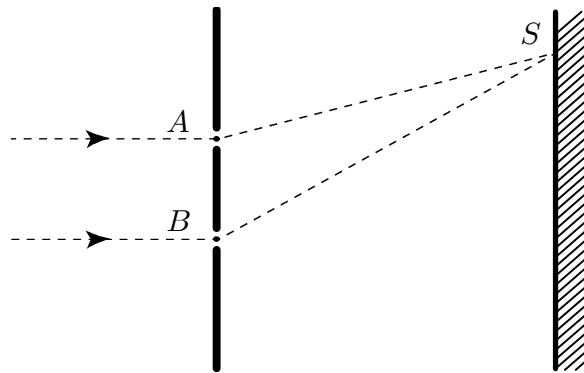

Figure 8.5.

The phase of the ray arriving at the screen passing through the first slit is different from that passing through the second slit, as the lengths of the two ray paths designated by *AS* and *BS* are different. The amplitude of the wave at the wall is equal to the sum of the amplitudes of the two rays, which interfere with each other.

If there are more than two slits, then there will be more rays interfering with one another. When we add up a large number of waves with the same frequency but different phases, the interference of the waves tends to cancel out one another. This is especially so as the frequency becomes very high.

To see this, let us consider the example of the integral

$$\int_0^b e^{i\lambda x}dx.$$

This integral is a superposition of waves of the same wave number λ but with a phase dependent on x. The magnitude of a wave is equal to unity when $0 < x < b$, and zero otherwise.

We easily carry out the integration and get

$$\int_0^b e^{i\lambda x}dx = \frac{e^{i\lambda b} - 1}{i\lambda}. \tag{8.33}$$

As we see from (8.33), this integral vanishes like $1/\lambda$ when λ is large. Since the absolute value of the integrand for $0 < x < b$ is unity, there are cancellations among the waves.

To understand how the cancellations come about, we observe that $e^{i\lambda x}$ is a rapidly varying function of x when λ is large. Indeed, as x varies by the small amount $1/\lambda$, the phase of $e^{i\lambda x}$ varies by the amount unity. Let us change the scale of the integration variable and define

$$t = \lambda x.$$

As x varies by the small amount $1/\lambda$, t varies by the amount unity. The integral in (8.33) becomes

$$I(\lambda) = \frac{1}{\lambda}\int_0^{b\lambda} e^{it}dt. \tag{8.34}$$

While the integrand of (8.24), considered as a function of x, is of the scale of $1/\lambda$, the integrand of

— 1 —

Asymptotic Expansions of Integrals

(8.34), considered as a function of t, is of the scale of unity, and is not a rapidly varying function. This is like looking at a photograph with tiny structures through a magnifier, and the small-scale variations become variations of normal scales. The factor $1/\lambda$ multiplying the integral in (8.34) is due to that fact that dx is smaller than dt by a multiplicative factor λ. The region of integration with t the integration variable is very large, and is from 0 to $b\lambda$. Yet as we integrate from $t = 0$ to $t = 2\pi n$, where n is any integer, we get exactly zero. This is because both the real part and the imaginary part of the function e^{it} are positive half of the time and negative half of the time, and the contributions of the waves integrated over a complete cycle of width 2π exactly cancel one another. Thus the region of uncancelled contributions in the t-variable is the region near the endpoint $t = \lambda b$ with a width less than 2π. If we put down the magnifier, we see that the uncancelled region in the x-variable is a region near the endpoint $x = b$ with a width of the order of $1/\lambda$. That is why the integral of (8.33) is of the order of $1/\lambda$.

Now let us change the lower limit of integration from zero to a, and consider the integral

$$\int_a^b e^{i\lambda x}dx,$$

It is clear that uncancelled contributions to the integral come from two small regions with widths of the order of $1/\lambda$ near the endpoints. The integral above can be exactly calculated. We have

$$\int_a^b e^{i\lambda x}dx = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda}. \tag{8.35}$$

Note that $e^{i\lambda b}$ and $e^{i\lambda a}$ are the values of the integrand at the endpoints b and a, respectively.