Lecture 18

In the image method, the "extra source" is ostensibly not on the boundary, it is on the other side of the boundary. However, we can transform it to what we want by the following trick: consider the function $u(\mathbf{x})$ in $\Omega=R^3$ that equals $(1/|\mathbf{x}-\mathbf{x}'| - 1/|\mathbf{x}-\mathbf{x}''|)/4\pi$ [the method-of-images solution] for $z>0$ and $u(\mathbf{x})=0$ for $z<0$. What right-hand-side does $-\nabla^2 u$ give? In $z>0$ $-\nabla^2 u$ gives $\delta(\mathbf{x}-\mathbf{x}')$ as before, and for $z<0$ $-\nabla^2 u$ gives zero. *At* $z=0$, however, there is a slope discontinuity in $(1/|\mathbf{x}-\mathbf{x}'| - 1/|\mathbf{x}-\mathbf{x}''|)/4\pi$, which means that $-\nabla^2 u$ also gives a $\delta(z)$ term: $\delta(z) \sigma(x,y)$ for a $\sigma(x,y)$ given by the amplitude of the slope discontinuity.

What does this mean? Our solution $u(\mathbf{x})$ is due to the sum of a point source at $\mathbf{x}'$ and *sources at the interface* ($z=0$). Worked out what these sources $\sigma(x,y)$ are. Physically, in the electrostatics example they correspond to a *surface charge density* on the surface of a conductor. Why are these sources there? They are there to *cancel* the effect of the source at $\mathbf{x}'$ for $z<0$, enforcing the boundary condition $u=0$ at $z=0$.

More generally, we can do this for *any* interface $d\Omega$: we can write the solution from a point source $\delta(\mathbf{x}-\mathbf{x}')$ in Ω as the sum of the solution from that point source plus an integral of *unknown point sources* $\sigma(\mathbf{x}')$ for points $\mathbf{x}'$ the boundary $d\Omega$. Formally, we determine $\sigma(\mathbf{x}')$ by requiring $u(\mathbf{x})$ to satisy the boundary condition at $d\Omega$, which gives a *surface integral equation* (SIE) (of the "first kind") for $\sigma(\mathbf{x}')$. Numerically, we discretize the surface in some way to get a finite number of unknowns approximating $\sigma(\mathbf{x}')$, leading to an SIE numerical method.

SIE methods (most commonly the "boundary element method", BEM) are very powerful in several ways. Because they only have unknowns on the *boundaries* (not everywhere in space like in a PDE method like finite differences), they can greatly reduce the number of unknowns: they handle the homogeneous regions analytically. They can handle infinite space (e.g. a cylinder surrounded by infinite space as in the pset) analytically, with no artificial truncation. Done properly, the matrices can have very nice properties. There are also some difficulties. SIE methods are not so efficient for problems that are not mostly homogeneous, especially continuously-varying media, and nonlinear or time-dependent problems are also difficult. Because the empty-space Green's function ($1/4\pi|\mathbf{x}-\mathbf{x}'|$ in 3d) blows up for nearby points, there are lots of tricky singularities that must be dealt with carefully in setting up SIE methods. Furthermore, because you have long-range interactions (every surface point interacts with every other point via the Green's function), the matrices are dense, not sparse. That means that developing fast solvers for large problems is tricky; remarkably, there are ways to do this (most famously the pioneering fast multipole method invented in 1985), but implementing them is not for the timid. Worse, the singularity-handling and fast-solver techniques depend heavily on the specific Green's function of empty space; for example, changing from 3d ($1/|\mathbf{x}-\mathbf{x}'|$) to 2d ($\ln|\mathbf{x}-\mathbf{x}'|$) problems requires a completely different implementation, although the concepts are similar.

Further reading: There are many books on integral-equation methods, e.g. *Boundary Integral Equation Methods for Solids and Fluids* by Bonnet is a reasonably general introduction.

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