18.303 Problem Set 3 Solutions

Problem 2: (5+5+5+10+5)

(a) Writing out the derivation in 18.02 fashion, this is tedious but straightforward:

$$\begin{aligned}
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} \\
&= \frac{\partial (u_y v_z - u_z v_y)}{\partial x} + \frac{\partial (u_z v_x - u_x v_z)}{\partial y} + \frac{\partial (u_x v_y - u_y v_x)}{\partial z} \\
&= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) v_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) v_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) v_z \\
&\quad u_x \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + u_y \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + u_z \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \\
&= (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v}).
\end{aligned}$$

(A much more compact derivation is possible using Einstein notation and the Levi–Civitá tensor, but probably most of you haven't seen this notation.)

(b) Given the above identity, integration by parts is straightforwards:

$$\begin{aligned}
\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) &= \int_\Omega \left[\nabla \cdot (\bar{\mathbf{u}} \times \mathbf{v}) + \overline{\nabla \times \mathbf{u}} \cdot \mathbf{v} \right] \\
&= \oiint_{\partial\Omega} (\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} + \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle,
\end{aligned}$$

applying the divergence theorem in the second line. So, the surface term $\oiint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$ is for $\boxed{\mathbf{w} = \bar{\mathbf{u}} \times \mathbf{v}}$.

(c) We must have $(\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} = 0$. Let $d\mathbf{S} = \mathbf{n}\, dS$, where $\mathbf{n}$ is the outward unit normal vector at each point on $\partial\Omega$. Then we must have $\bar{\mathbf{u}} \times \mathbf{v} \perp \mathbf{n}$, which is true if, for example, both $\mathbf{u}$ and $\mathbf{v}$ are parallel to $\mathbf{n}$ at the boundary. i.e. if $\boxed{\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0}$. It is *not* necessary to require $\mathbf{u}|_{\partial\Omega} = 0$ on the boundary, and that answer will not be accepted as I specifically requested that you not constrain all the components of $\mathbf{u}$ on the boundary.

Another way to see this is to write $(\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} = \mathbf{n} \cdot (\bar{\mathbf{u}} \times \mathbf{v}) dS = \mathbf{v} \cdot (\mathbf{n} \times \bar{\mathbf{u}}) dS = \bar{\mathbf{u}} \cdot (\mathbf{v} \times \mathbf{n}) dS$ by elementary triple-product identities, and hence we again see that it is sufficient to have $\mathbf{u} \times \mathbf{n} = 0$ on the boundary.

Although I will accept the above answer, it is actually possible to contrive a slightly weaker condition: $\mathbf{u}$ and $\mathbf{v}$ can have components $\perp$ to $\mathbf{n}$ on the boundary, as long as those surface-parallel components are *in the same direction* for both $\mathbf{u}$ and $\mathbf{v}$ (to obtain zero cross product). That is, suppose $\mathbf{p}(\mathbf{x})$ is some surface-parallel ($\mathbf{p} \perp \mathbf{n}$) vector field on $\partial\Omega$.[1] Then it is sufficient for the surface-parallel component of $\mathbf{u}$ to be $\perp$ to $\mathbf{p}$ everywhere on the boundary, or equivalently $\mathbf{u} \cdot \mathbf{p}|_{\partial\Omega} = 0$.

[Actually, there are other possible conditions on $\mathbf{u}$ if we allow non-local boundary conditions, where $\mathbf{u}$ at one point on the boundary is related to $\mathbf{u}$ at another point. For example, if Ω is a cube and $\mathbf{u}$ is *periodic* (i.e. $\mathbf{u}$ on one face equals $\mathbf{u}$ on the opposite face), then the

[1] By the "hairy ball theorem" of topology, $\mathbf{p}$ must vanish somewhere on $\partial\Omega$ if Ω is simply connected (no holes), at which point $\mathbf{u}$ must be parallel to $\mathbf{n}$.

$\oiint_{\partial\Omega}$ vanishes because each face of the cube cancels the opposite face, without requiring any component of $\mathbf{u}$ to be zero.]

(d) We just "integrate by parts" twice:

$$\begin{aligned}\langle \mathbf{u}, \nabla\times\nabla\times\mathbf{v}\rangle &= \int_\Omega \bar{\mathbf{u}}\cdot(\nabla\times\nabla\times\mathbf{v}) = \cancel{\oiint_{\partial\Omega}[\bar{\mathbf{u}}\times(\nabla\times\mathbf{v})]}\cdot d\mathbf{S} + \int_\Omega (\nabla\times\bar{\mathbf{u}})\cdot(\nabla\times\mathbf{v}) \\ &= \cancel{\oiint_{\partial\Omega}[(\nabla\times\bar{\mathbf{u}})\times\mathbf{v}]}\cdot d\mathbf{S} + \int_\Omega (\nabla\times\nabla\times\bar{\mathbf{u}})\cdot\mathbf{v} = \langle\nabla\times\nabla\times\mathbf{u},\mathbf{v}\rangle,\end{aligned}$$

where the surface terms cancel if *either* $\boxed{\mathbf{u}\times\mathbf{n}|_{\partial\Omega}=0}$ or $\boxed{(\nabla\times\mathbf{u})\times\mathbf{n}|_{\partial\Omega}=0}$, that is if either $\mathbf{u}$ or its curl are normal to the surface. (As in the previous part, one can actually weaken this slightly, but this is sufficient for our purposes.)

To check definiteness, carry integration by parts "halfway" through:

$$\langle\mathbf{u},\nabla\times\nabla\times\mathbf{u}\rangle = \int_\Omega |\nabla\times\mathbf{u}|^2 \geq 0,$$

so it is **positive semidefinite**. It is *not* positive-definite since $\nabla\times\mathbf{u}=0$ for $\mathbf{u}=\nabla\phi\neq 0$ for *any* non-constant scalar field ϕ , and we can easily choose such a ϕ such that $\nabla\phi$ satisfies the boundary conditions (e.g., choose ϕ so that it is constant in a neighborhood of $\partial\Omega$).

(e) Taking the curl of both sides of $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, we obtain $\nabla\times\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} = -\frac{\partial^2\mathbf{E}}{\partial t^2}$. That is, we have

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = \hat{A}\mathbf{E}$$

where $\hat{A} = -\nabla\times\nabla\times$. From the previous parts, for $\mathbf{E}\perp\mathbf{n}$ at the surface, this $\hat{A}$ is self-adjoint and negative semidefinite, and hence we have a **hyperbolic equation**.

As in class, we therefore expect orthogonal eigenfunctions and real $\lambda\leq 0$, and hence oscillating "normal mode" solutions with eigenfrequencies $\omega=\sqrt{-\lambda}$.

[Technically, we also obtain $\lambda=0$ solutions which are non-oscillatory—from above, these are nullspace solutions $\mathbf{E}=\nabla\phi$ for some ϕ, which physically correspond to the time-independent fields of fix charge distributions, where $-\phi$ is the potential and $\nabla\cdot\mathbf{E}=\nabla^2\phi$ is the charge density. Everything else, all of the other eigenfunctions, are oscillating solutions.]

Problem 2: (5+5+5+5+10)

See also the IJulia notebook posted with the solutions.

(a) Setting the slopes to be zero at R_1 and R_2 simply gives

$$\alpha J_m'(kr) + \beta Y_m'(kr) = 0$$

at the two radii, or $E\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ where

$$\boxed{E = \begin{pmatrix} J_m'(kR_1) & Y_m'(kR_1) \\ J_m'(kR_2) & Y_m'(kR_2)\end{pmatrix}}.$$

Hence, writing $f_m(k) = \det E$, we get

$$\boxed{f_m(k) = J_m'(kR_1)Y_m'(kR_2) - J_m'(kR_2)Y_m'(kR_1)}.$$

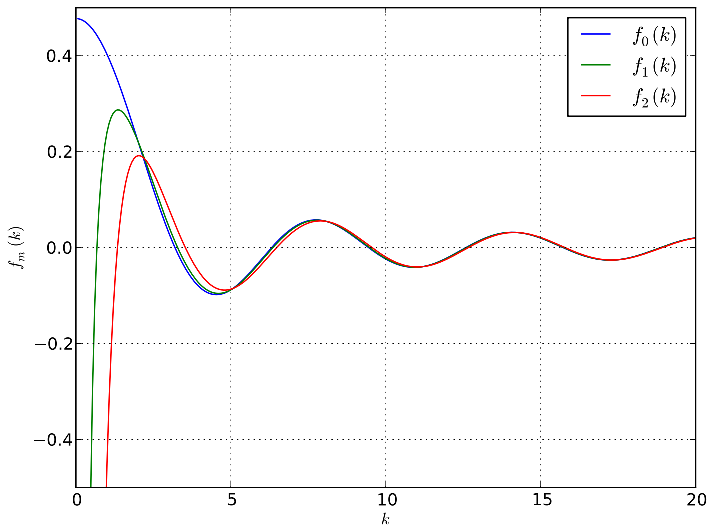

Figure 1: Plot of $f_m(k)$ for $m = 0, 1, 2$.

Given a k for which $f_m(k) = 0$, then we can solve for the nullspace of E by arbitrarily choosing a scaling such that $\boxed{\alpha = 1}$ and solving for β from the first *or* second rows of $E\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$:

$$\boxed{\beta = -\frac{J_m'(kR_1)}{Y_m'(kR_1)} = -\frac{J_m'(kR_2)}{Y_m'(kR_2)}}.$$

(b) The plot is shown in Figure 1. Note that $f_m(k)$ for $m > 0$ has a divergence as $k \to 0$, so we used the `ylim` command to rescale the vertical axis (otherwise it would be hard to read the plot!); see the solution IJulia notebook.

(c) We'll use the Scilab `newton` function, similar to class, to find the roots, with initial guesses provided by our plot in Figure 1. We find $k_1 \approx 3.196578$, $k_2 \approx 6.31234951$, and $k_3 \approx 9.4444649$. See the solutions notebook.

(d) See the IJulia notebook. Using our k_1 and k_2 from part (c) and our α and β from part (a), we find that $\int_{R_1}^{R_2} r u_{0,1}(r) u_{0,2}(r) dr \approx 10^{-15}$, which is zero up to roundoff errors.

(e) Here, we have $\hat{A} = c\nabla^2$, and obtain:

(i) At the origin, we can't blow up, and therefore we only have J for $r < R_1$, but we have both J and Y outside this. Hence

$$u(r,\theta) = [A\cos(m\theta) + B\sin(m\theta)] \times \begin{cases} \alpha J_m(k_1 r) & r < R_1 \\ \beta J_m(k_2 r) + \gamma Y_m(k_2 r) & r > R_1 \end{cases}.$$

Note that the angular dependence must be the same for all r in order to have continuity at $r = R_1$. However, we are not guaranteed that $k_1 = k_2$! In particular, we want $\hat{A}u = \lambda u$ for some λ, and plugging in the Bessel functions and the values of c we find $\lambda = -2k_1^2 = -k_2^2$. Hence, we let $k_2 = k$ and $k_1 = k/\sqrt{2}$.

(ii) To get a finite $\hat{A}u$, we must have u and $\partial u/\partial r$ continuous at $r = R_1$. Combined with $u = 0$ at $r = R_2$, this gives the equations

$$\begin{bmatrix} -J_m(kR_1/\sqrt{2}) & J_m(kR_1) & Y_m(kR_1) \\ -J'_m(kR_1/\sqrt{2}) & J'_m(kR_1) & Y'_m(kR_1) \\ 0 & J_m(kR_2) & Y_m(kR_2) \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E_m(k) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0,$$

where the first two rows are the continuity conditions and the last row is the Dirichlet condition, and we have defined a matrix $E_m(k)$.

(iii) We simply need $f_m(k) = \det E_m(k)$ to solve for the solutions k and hence the eigenvalues.