Lecture 31

6.3 Examples of Manifolds

We begin with a review of the definition of a manifold.

Let X be a subset of $\mathbb{R}^n$, let Y be a subset of $\mathbb{R}^m$, and let $f : X \to Y$ be a continuous map.

Definition 6.6. The map f is $\mathcal{C}^\infty$ if for every $p \in X$, there exists a neighborhood U_p of p in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ map $g_p : U_p \to \mathbb{R}^m$ such that $g_p = f$ on $U_p \cap X$.

Claim. *If $f : X \to Y$ is continuous, then there exists a neighborhood U of X in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ map $g : U \to \mathbb{R}^m$ such that $g = f$ on $U \cap X$.*

Definition 6.7. The map $f : X \to Y$ is a *diffeomorphism* if it is one-to-one, onto, and both f and f^{-1} are $\mathcal{C}^\infty$ maps.

We define the notion of a manifold.

Definition 6.8. A subset X of $\mathbb{R}^N$ is an *n-dimensional manifold* if for every $p \in X$, there exists a neighborhood V of p in $\mathbb{R}^N$, an open set U in $\mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$.

Intuitively, the set X is an n-dimensional manifold if locally near every point $p \in X$, the set X "looks like an open subset of $\mathbb{R}^n$."

Manifolds come up in practical applications as follows:

Let U be an open subset of $\mathbb{R}^N$, let $k < N$, and let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. Suppose that 0 is a regular value of f, that is, $f^{-1}(0) \cap C_f = \phi$.

Theorem 6.9. *The set $X = f^{-1}(0)$ is an n-dimensional manifold, where $n = N - k$.*

Proof. If $p \in f^{-1}(0)$, then $p \notin C_f$. So the map $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ is onto. The map f is a submersion at p.

By the canonical submersion theorem, there exists a neighborhood V of 0 in $\mathbb{R}^n$, a neighborhood U_0 of p in U, and a diffeomorphism $g : V \to U$ such that

$$f \circ g = \pi. \tag{6.7}$$

Recall that $\mathbb{R}^N = \mathbb{R}^\ell \times \mathbb{R}^n$ and $\pi : \mathbb{R}^N \to \mathbb{R}^k$ is the map that sends

$$(x, y) \in \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k. \tag{6.8}$$

Hence, $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n = \mathbb{R}^n$. By Equation 6.7, the function g maps $V \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. But $V \cap \pi^{-1}(0)$ is a neighborhood of 0 in $\mathbb{R}^n$ and $U_0 \cap f^{-1}(0)$ is a neighborhood of p in X. $\square$

We give three examples of applications of the preceding theorem.

1. We consider the n-sphere S^n. Define a map

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad f(x) = x_1^2 + \ldots + x_{n+1}^2 - 1. \tag{6.9}$$

   The derivative is $(Df)(x) = 2[x_1, \ldots, x_{n+1}]$, so $C_f = \{0\}$. If $a \in f^{-1}(0)$, then $\sum a_i^2 = 1$, so $a \notin C_f$. Thus, the set $f^{-1}(0) = S^n$ is an n-dimensional manifold.

2. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. Define

$$X = \text{graph } g = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : y = g(x)\}. \tag{6.10}$$

   Note that $X \subseteq \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$.

   Claim. *The set X is an n-dimensional manifold.*

   Proof. Define a map $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ by

$$f(x, y) = y - g(x). \tag{6.11}$$

   Note that $Df(x, y) = [-Dg(x), I_k]$. This is always of rank k, so $C_f = \phi$. Hence, the graph g is an n-dimensional manifold. □

3. The following example comes from Munkres section 24, exercise #6. Let

$$\mathcal{M}_n = \text{ the set of all } n \times n \text{ matrices,} \tag{6.12}$$

   so

$$\mathcal{M}_n \cong \mathbb{R}^{n^2}. \tag{6.13}$$

   With any element $[a_{ij}]$ in $\mathcal{M}_n$ we associate a vector

$$(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots). \tag{6.14}$$

   Now, let

$$\mathcal{S}_n = \{A \in \mathcal{M}_n : A = A^t\}, \tag{6.15}$$

   so

$$\mathcal{S}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}}. \tag{6.16}$$

   With any element $[a_{ij}]$ in $\mathcal{S}_n$ we associate a vector

$$(a_{11}, \ldots, a_{1n}, a_{22}, a_{23}, \ldots, a_{2n}, a_{33}, a_{34}, \ldots). \tag{6.17}$$

   The above association avoids the "redundancies" $a_{12} = a_{21}, a_{31} = a_{13}, a_{32} = a_{23}$, etc.

   Define

$$O(n) = \{A \in \mathcal{M}_n : A^t A = I\}, \tag{6.18}$$

   which is the set of orthogonal $n \times n$ matrices.

   As an exercise, the student should prove the following claim.

Claim. *The set $O(n) \subseteq \mathcal{M}_n$ is an $\frac{n(n-1)}{2}$-dimensional manifold.*

Proof Hint: First hint: Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map defined by

$$f(A) = A^t A - I, \tag{6.19}$$

so $O(n) = f^{-1}(0)$. Show that $f^{-1}(0) \cap C_f = \phi$. The main idea is to show that if $A \notin f^{-1}(0)$, then the map $Df(A) : \mathcal{M}_n \to \mathcal{S}_n$ is onto.

Second hint: Note that $Df(A)$ is the map the sends $B \in \mathcal{M}_n$ to $A^t B + B^t A$. $\square$

Manifolds are often defined by systems of non-linear equations:

Let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a continuous map, and suppose that $C_f \cap f^{-1}(0) = \phi$. Then $X = f^{-1}(0)$ is an n-dimensional manifold. Suppose that $f = (f_1, \ldots, f_k)$. Then X is defined by the system of equations

$$f_i(x_1, \ldots, x_N) = 0, \quad i = 1, \ldots, k. \tag{6.20}$$

This system of equations is called *non-degenerate*, since for every $x \in X$ the matrix

$$\left[\frac{\partial f_i}{\partial x_j}(x) \right] \tag{6.21}$$

is of rank k.

Claim. *Every n-dimensional manifold $X \subseteq \mathbb{R}^N$ can be described locally by a system of k non-degenerate equations of the type above.*

Proof Idea: Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold. Let $p \in X$, let U be an open subset of $\mathbb{R}^n$, and let V be an open neighborhood of p in $\mathbb{R}^N$. Let $\phi : I \to V \cap X$ be a diffeomorphism. Modifying ϕ by a translation if necessary we can assume that $0 \in U$ and $\phi(0) = p$. We can think of ϕ as a map $\phi : U \to \mathbb{R}^N$ mapping U into X.

Claim. *The linear map $(D\phi)(0) : \mathbb{R}^n \to \mathbb{R}^N$ is injective.*

Proof. The map $\phi^{-1} : V \cap X \to U$ is a $\mathcal{C}^\infty$ map, so (shrinking V if necessary) we can assume there is a $\mathcal{C}^\infty$ map $\psi : V \to U$ with $\psi = \phi^{-1}$ on $V \cap X$. Since ϕ maps U onto $V \cap X$, we have $\psi \circ \phi = \phi^{-1} \circ \phi = I =$ the identity map of U onto itself. Thus,

$$I = D(\psi \circ \phi)(0) = (D\psi)(p)(D\phi)(0). \tag{6.22}$$

That is, $D\psi(p)$ is a "left inverse" of $D\phi(0)$. So, $D\phi(0)$ is injective. $\square$

We can conclude that $\phi : U \to \mathbb{R}^N$ is an immersion at 0. The canonical immersion theorem tells us that there exists a neighborhood U_0 of 0 in U, a neighborhood V_p of p in V, and a $\mathcal{C}^\infty$ map $g : V_p \to \mathbb{R}^N$ mapping p onto 0 and mapping V_p diffeomorphically onto a neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^N$ such that

$$\iota^{-1}(\mathcal{O}) = U_0 \tag{6.23}$$

and

$$g \circ \phi = \iota \tag{6.24}$$

on U_0. Here, the map ι is the canonical submersion map $\iota : \mathbb{R}^n \to \mathbb{R}^N$ that maps $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 0, \dots, 0)$.

By Equation 6.24, the function g maps $\phi(U_0)$ onto $\iota(U_0)$. However, by Equation 6.23, the set $\iota(U_0)$ is the subset of $\mathcal{O}$ defined by the equations

$$x_i = 0, \quad i = n+1, \dots, N. \tag{6.25}$$

So, if $g = (g_1, \dots, g_N)$, then $\phi(U_0) = X \cap V_p$ is defined by the equations

$$g_i = 0, \quad i = n+1, \dots, N. \tag{6.26}$$

Moreover, the $N \times N$ matrix

$$\left[\frac{\partial g_i}{\partial x_j}(x)\right] \tag{6.27}$$

is of rank N at every point $x \in V_p$, since $g : V_p \to \mathcal{O}$ is a diffeomorphism. Hence, the last $N - n$ row vectors of this matrix

$$\left(\frac{\partial g_i}{\partial x_1}, \dots, \frac{\partial g_i}{\partial x_N}\right), \quad i = n+1, \dots, N, \tag{6.28}$$

are linearly independent at every point $x \in V_p$.

Now let $k = N - n$ and let $f_i = g_{i+n}$, $i = 1, \dots, k$. Then $X \cap V_p$ is defined by the equations

$$f_i(x) = 0, \quad i = 1, \dots, k, \tag{6.29}$$

and the $k \times N$ matrix

$$\left[\frac{\partial f_i}{\partial x_k}(x)\right] \tag{6.30}$$

is of rank k at all points $x \in V_p$. In other words, the system of equations 6.29 is non-degenerate. $\square$