18.100B Fall 2010
Practice Quiz 1 Solutions

1.(a) $E \subset X$ is compact if, given any open cover $E \subset \bigcup_{\alpha \in A} U_\alpha$ by open sets $U_\alpha \subset X$ (with A any index set), one can find a finite subcover $E \subset \bigcup_{\alpha \in A'} U_\alpha$, with $A' \subset A$ a finite subset.

(b) $E = \{e_1, \ldots, e_N\}$
Given an open cover $E \subset \bigcup_{\alpha \in A} U_\alpha$, for $i = 1, \ldots, N$ pick $\alpha_i \in A$ s.t. $e_i \in U_{\alpha_i}$, then $A' := \{\alpha_1, \ldots, \alpha_N\} \subset A$ is finite and $E \subset \bigcup_{\alpha \in A'} U_\alpha$ is still a cover.

(c) $E = \mathbb{N} \subset \mathbb{R}$
(because $\mathbb{N} \subset \bigcup_{n \in \mathbb{N}} B_{1/2}(n)$ is an open cover with $m \in B_{1/2}(n) \Rightarrow m = n$, so if
$\mathbb{N} \subset \bigcup_{n \in A'} B_{1/2}(n)$ then necessarily $m \in A' \;\; \forall m \Rightarrow A' = \mathbb{N}$ infinite)

2.(a) By assumption we have $f : \mathbb{N} \to A, g : \mathbb{N} \to B$ bijections.
Define a surjection $h : \mathbb{N} \to A \cup B$ by $h(2n-1) = f(n), h(2n) = g(n)$.
Then we can make it a bijection $h' : \mathbb{N} \to A \cup B$ by $h'(1) = h(1), h'(n+1) = h(m_n)$ with $m_n := \min\{k \in \mathbb{N} | h(k) \notin \{h'(1), \ldots, h'(n)\}\}$
(m always exists because A infinite $\Rightarrow A \cup B$ infinite).
Similarly, define $f'(n) := f(m_n)$ with $m_n := \min\{k \in \mathbb{N} | f(k) \in A \cap B \backslash \{f(1), \ldots, f(n)\}\}$.
If for some $n \in \mathbb{N}, A \cap B \backslash \{f(1), \ldots, f(n)\} = \emptyset$, then $A \cap B$ is finite ($\sim \{1, \ldots, n\}$); otherwise this defines a bijection $\mathbb{N} \to A \cap B$, so $A \cap B$ is countable.

(b) By definition,
$$\left.\begin{array}{l} \bullet \forall s \in S \quad s \leq \sup S \\ \bullet \forall t \in T \quad t \leq \sup T \end{array}\right\} \Rightarrow \forall s + t \in S + T \qquad s + t \leq \sup S + \sup T$$
So $\sup S + \sup T$ is an upper bound.
$\bullet \forall \gamma < \sup S \quad \gamma$ is not an upper bound, i.e. $\exists s \in S : \gamma < s$
$\bullet \forall \beta < \sup T \quad \beta$ is not an upper bound, i.e. $\exists t \in T : \beta < t$
$\Rightarrow$ Given $\alpha < \sup S + \sup T$ write $\alpha = \gamma + \beta, \gamma < \sup S, \beta < \sup T$
$$\left(\begin{array}{l} \gamma = \sup S - \frac{1}{2}(\sup S + \sup T - \alpha) \\ \beta = \sup T - \frac{1}{2}(\sup S + \sup T - \alpha) \end{array} \right)$$
then $\exists s \in S, t \in T : \alpha = \gamma + \beta < s + t$,
$$\Downarrow$$
$$s + t \in S + T$$
so α is not an upper bound.

3.(a) Only 0

(Because $B_r(0) \cap E$ for $r > 0$ always contains some $\frac{1}{n} \neq 0$. All other points are isolated, $B_{1/2n}(\frac{1}{n}) \cap E = \{\frac{1}{n}\}$)

(b)

$\rightarrow$ Finite subsets (which never have limit points)

$\rightarrow$ Infinite subsets that contain 0

(c) Finite subsets and infinite subsets that contain 0

Because $X \subset \mathbb{R}$ is compact (bounded and closed) and the compact subsets of a compact set are exactly the closed subsets.

4.(a) FALSE

$\mathrm{int}(\bar{A})$ does not contain isolated points of A

(b) FALSE

Not definite: e.g. $f(x) = x, g(x) = x^2, d(f,g) = |0 - 0| = 0$ but $f \neq g$.

(c) TRUE

Because closed subsets of compact sets are compact

(d) FALSE

There is an uncountable subset $\{(x, -x) | x \in \mathbb{R}\} \simeq \mathbb{R}$

(e) TRUE

$$\left.\begin{array}{l} x + y \in \mathbb{Q} \\ x - y \in \mathbb{Q} \end{array}\right\} \Rightarrow x, y \in \mathbb{Q}, \text{ so the set is } \mathbb{Q} \times \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \times \mathbb{Q},$$

which is countable as a countable union of countable sets.

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