MARTINA BALAGOVIC: Welcome.

Today's problem actually appeared in a quiz. It appeared in quiz one in fall of 1999 as question four. The problem puts the usual solve the following system upside down by saying we have some matrix and we know that all the solutions to A*x equals this vector here, [1, 4, 1, 1], all the solutions to this problem are given by x equals [0, 1, 1] plus any number c times [1, 2, 1]. And we're asked to say everything that we can about the columns of the matrix A.

So I'm going to let you pretend that you are on an exam, try to solve it yourself, and then come back and compare your solution with mine.

OK, welcome back. So the first thing that you should think about in this sort of situation is what is the size of A? Well, we want to multiply A with an x that has three entries, so A should have three columns. Let me call those columns c_1, c_2, and c_3. And when I take some linear combinations of c_1, c_2 and c_3, I'm going to get this vector here, [1, 4, 1, 1]. So all the c_i's, c_1, c_2, and c_3 are vectors in R_4.

Now, if you know about, if you learned about particular solutions and special solutions, then my notation here shouldn't surprise you. I'm going to call this vector here x_p, and this vector here x_s. And I'm going to use the fact that x_p plus c times x_s satisfies A times this equals b-- I will call this vector b-- for any number c.

In particular, what I'm going to conclude is that when c equals 0 we get A times x_p equals b. But also that when c equals 1, we get A times x_p plus A times x_s equals b. Replacing this by b, we get that this implies that A times x_s equals 0.

So in trying to find what are the columns c_1, c_2, and c_3 of the matrix A, let's look at these two equations. x_p satisfies A times x_p equals b, and x_s satisfies A times x_s equals 0. Again, if you know what particular and special solutions are this shouldn't surprise you. But we also know what x_p and x_s are, so let's use them to try to calculate c_1, c_2, and c_3.

A times x_p equals b means that the linear combination of c_1, c_2, and c_3 encoded in the vector x_p, which is [0, 1, 1], gives the vector b. So c_1, c_2, c_3 times [0, 1, 1] gives us [1, 4, 1, 1]. In other words, c_2 plus c_3 equal b.

Let's turn our attention to A times x_s equals 0. This says that c_1, c_2, c_3 times-- x_s was defined to be [0, 2, 1]-- equals 0. In other words, 2 times c_2 plus c_3 equals 0.

Now solving this system where the unknowns are vectors but it's still just a linear system, we can see, for example, from the second equation that c_3 equals minus 2*c_2. And plugging it back into the original equation, getting c_2 minus 2*c_2 equals b, from which it follows that c_2 is equal to minus b, and that c3 is equal to 2 times b.

So from this tiny amount of information-- we just knew the solutions to this one particular equation involving A-- we got the second column of A and the third column of A completely explicitly calculated.

Now, what can we say about the first column? I said before that all the solutions of A*x equals b are of the form a particular solution plus some number times a special solution. And the information that we have is that there's just one number here. So we said everything, once we remove this vector here, everything that we get here will satisfy A times x equals 0.

And the fact that everything that satisfies A times x equals 0 is a multiple of this one vector that was given to us means that the null space of A has dimension one. There is just one special solution. So dimension of the null space of A is 1. So rank of A is the number of columns minus this dimension of null space, and it's equal to 2.

As rank of A is equal to 2, the number of linearly independent columns needs to be 2 as well. So the only thing that we can say at this point is if the first column was also a multiple of b, as the second and the third are, then the rank would be smaller than 2. So that is the only thing that cannot happen. So c_1 is not a multiple of b. Not any multiple, including not a zero multiple.

And that's pretty much everything we can say. Yes, if it was some other multiple of it, then we would be able to find some other vector here and we would have two parameters. But it's not, and this is everything that we can say about it.