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Hi everyone. Welcome back.

SHIROKOFF:

So today we're going to tackle a problem in complex matrices. And specifically, we're going to look at diagonalizing a complex matrix. So given this matrix A , we're asked to find its eigenvalue matrix λ , and its eigenvector matrix S .

And one thing to note about this matrix A is that if we take its conjugate transpose, it's actually equal to itself. So in Professor Strang's book, he combines this notation to be superscript H to mean conjugate transpose. So if you were to take the transpose of this matrix and then conjugate all the elements, you would find that A equals its conjugate transpose, and we call this property Hermitian.

So I'll let you think about this for a moment and I'll be back in a second.

OK, welcome back. So what's the first step in computing the eigenvectors and eigenvalues of the matrix is to take a look at the characteristic equation. So specifically, we take $\det(A - \lambda I)$. And quite possibly, the only thing new with this problem is that the entries of the matrix A are complex.

Now, you may have already seen that λ 's being complex, but we're going to work this out explicitly. So if I take the determinant, we get $\det \begin{pmatrix} 2 - \lambda & 3 - \lambda \\ 1 - i & 1 + i \end{pmatrix}$. We want to set this to 0. This then gives us a polynomial for λ . $(2 - \lambda)(1 + i) - (3 - \lambda)(1 - i) = 0$. We can expand out this term. You get $6 - 5\lambda + \lambda^2 = 0$.

These two terms you'll note are complex conjugates of each other. This tends to make things simple. So we have $(-1 - i)^2 = 2$. Because they're differences of squares, the cross terms involving i cancel, and we get the characteristic equation. $\lambda^2 - 5\lambda + 4 = 0$.

And specifically, we can factorize this equation. We see that there's roots of -1 , or factorizes into factors of $\lambda - 1$ and $\lambda - 4$, which then give us roots of

$\lambda = 1$ and $\lambda = 4$.

So when one curious point to note is that the eigenvalues are real in this case. 1 and 4 are real, whereas the matrix that we started with was complex. And this is a general property of Hermitian matrices. So even though they might be complex matrices, Hermitian matrices always have real eigenvalues.

So this is the first step when asked to diagonalize a matrix. The second step is to find the eigenvectors. And to do that what we have to do is we have to look at the cases for $\lambda = 1$ and $\lambda = 4$ separately. So let's first look at the case of $\lambda = 1$. And specifically, we're going to be looking for a vector such that $(A - \lambda I)v = 0$. And if we've done things properly, this matrix $A - \lambda I$ should be singular.

So if we take $A - \lambda I$, we're going to get $\begin{pmatrix} 1 - i & 1 + i \\ 1 + i & 3 - 1 \end{pmatrix}$ is $\begin{pmatrix} 1 - i & 1 + i \\ 1 + i & 2 \end{pmatrix}$. And I'll write out components of v , which are v_1 and v_2 . And we want this to be 0 . And you'll note that it's almost always the case that when we work out $A - \lambda I$, the second row is going to be a constant multiple of the first row. And this must be the case because these two rows must be linearly dependent on each other for the matrix $A - \lambda I$ to be singular.

So if you look at this you might think that these two rows aren't necessarily linearly independent. But the point is that there's complex numbers involved. And indeed, actually if we were to multiply this first row by $1 + \lambda$, we would get $1 + \lambda$ and 2 . And you note that that's exactly the second row. So this second row is actually $1 + \lambda$ times the first row. So these rows are actually linearly dependent on each other.

So what values of v_1 and v_2 can we take? Well we just need to make this top row multiplied by v_1 and v_2 equal to 0 . And then because the second row is a constant multiple of the first row, we're automatically guaranteed that the second equation holds.

So just by looking at it, I'm going to take $v_1 = 1 - i$, and v_2 is negative

1. So we see that $1 - i - 1 + i$ is going to give us 0. So this is one solution. And of course, we can take any constant multiple times this eigenvector, and that's also going to be an eigenvector.

So I'll just write this out. $1 - i, -1$ is the eigenvector for $\lambda = 1$. For $\lambda = 4$, again, $A - \lambda I$ is going to give us negative 2, $1 - i, 1 + i, 3 - \lambda$ going to be minus 1. And I'll call this vector u_1 and u_2 . And again, we want u_1 and u_2 equal to 0-- or sorry, the matrix multiplied by u_1, u_2 is equal to 0.

And just by looking at this again, we see that the second row is actually a constant multiple of the first row. For example, if we were to multiply this row by negative 2, and this row by $1 + i$, we would see that they're constant multiples of each other. So I can take u to be, for example, $1, 1 + i$. How did I get this? Well I just looked at the second equation because it's a little simpler, and I said well if I have $1 + i$ here, I can just say multiply it by 1. And then $-1 + i$ when I add them up is going to vanish. So this is how I get the second one.

Now there's something curious going on, and this is going to be another property of Hermitian matrices. But if you actually take a look at this eigenvector, it will be orthogonal to this eigenvector when we conjugate the elements and dot the two vectors together. So this is another very special property of complex Hermitian matrices.

OK, so the last step now is to construct these matrices λ and S . Now we already know what λ is because it's the diagonal matrix with the eigenvalues 1 and 4. So we have $1, 0, 0$ and 4. Now I'm going to do something special for S . I've noted that these two vectors u and v are orthogonal to each other.

So what do I mean by orthogonal? Specifically, if I were to take v conjugate transpose and multiply it by u , we would end up getting $1 + i - 1$. This would be v conjugate transpose $\cdot u$. $1, 1 + i$, and we see that when we multiply these out we get 0.

So when we have orthogonal eigenvectors, there's a trick that we can do to build up this matrix S and S inverse. What we can do is we can normalize u and v . So specifically, we can take any constant multiple of u and v , and it's still going to be an eigenvector.

So what I'm going to do is I'm going to take u and v and multiply them by their length. So for example u , the amplitude of its top component is 1. The amplitude of its bottom component is 2. So notice that the modulus of the complex number $1 + i$ is $\sqrt{2}$. So we have-- sorry, it's $\sqrt{2}$, the complex modulus is $\sqrt{2}$. So the amplitude of the entire vector is $\sqrt{3}$. It's $1 + 2$ squared quantity squared rooted, so it's $\sqrt{3}$.

So what we can do is we can build up this matrix S using a normalization factor of $1/\sqrt{3}$. And I'm going to take the first column is the first eigenvector that corresponds to eigenvalue 1. And then the second column is the second eigenvector which corresponds value 4. And the reason I put in this $\sqrt{3}$ here is to make this column unit length 1, and this column unit length 1.

And the reason I do this is because now this matrix S , it's possible to check that this matrix S is actually unitary, which means that its inverse is actually just equal to its conjugate transpose. So this is a very special property of the eigenvectors of a Hermitian matrix.

And then lastly, I'm just going to write down the diagonalization of A . So if I have A , because I have its eigenvector matrix S , and its eigenvalue matrix λ , it's possible to decompose A into a product of $S \lambda S^{-1}$. And because S is unitary, its inverse is actually its conjugate transpose.

So just to put the pieces together, we have A is equal to $S \lambda S^{-1}$, which is $1/\sqrt{3}$ times $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ times S^{-1} , which is going to be its conjugate transpose. So what I do is I conjugate each element, so $1 - i$ becomes $1 + i$ and vice versa. And then I take the transpose. So I get $1 + i$. Transposing swaps the -1 and 1 . And at the end of the day, I get S^{-1} inverse is just this matrix here.

And if you were to multiply these matrices out, you would see it you actually do recover A .

So just to summarize quickly, even though we were given a complex matrix A , the process to diagonalize A is the same as what we've seen before. The first step is to find the characteristic equation and the eigenvalues. And then the second step is to find the eigenvectors, and you do this in the same procedure. But in general, the eigenvectors can be complex. And for this very special case, when A is Hermitian, the eigenvalues are real, and the eigenvectors are orthogonal to each other.

So I think I'll conclude here, and I'll see you next time.