

**ANA RITA PIRES:** Hi. Welcome back to recitation. In lecture, you've been learning about the properties of determinants. To remember, there were three main properties, and then seven more that fall out of those three. I'll tell you what these three were.

The first one was the determinant of the identity matrix is always equal to 1. If you switch two rows in a matrix, the determinant switches sign, and the determinant is a function of each-- it's a linear function of each row separately. And there's seven more. We'll use them here.

Today's problem is about finding the determinants of matrices by using these properties. So here you have four matrices. A has lots of 100s, 200s, and 300s numbers. B is called a Vandermonde matrix. It has a very nice structure with 1s, and then  $abc$ ,  $a^2$ ,  $b^2$ ,  $c^2$ . It can be bigger, and you'll just have cubes and more letters down here. C is given by the product of these two, and D is this matrix.

Good luck. Hit pause, work on them, and when you're ready, come back and I'll show you how I did it.

Did you get some? OK, let's do it. Starting with matrix A. I have lots of big numbers. I suggest that we do a little bit of elimination, because as you know, doing elimination steps, except for permuting rows, doesn't change the determinant in the matrix.

So let's do determinant of A is equal to 101. 201. 301. Then if I subtract off the first row from the second one, I'll get 1, 1, 1, which is very convenient. And actually, if you subtract the second row from the third one, you'll get 1, 1, 1. Here's the property of the determinant. If you have two equal rows on your matrix, the determinant is automatically equal to zero. All right. All done with one of them. Let's work on the second one.

The determinant of B. Well let's try elimination again.  $1, a, a^2$ .  $1 - 1$  is 0,  $b - a$ , and  $b^2 - a^2$ .  $b^2 - a^2$ -- let me

factor that into  $b - a$ ,  $b + a$ , which will be very convenient in the next step. And then I'm going to subtract the first row again from the third one. I'll get  $0, c - a$ . And again, I'll get  $c^2 - a^2, c - a, c + a$ .

Let's use that third property that was the determinant is linear on each row separately. So what I'm going to do is, see this factor of  $b - a$ ? It shows up in every entry of this row. Well it's a  $0$ , so it's the  $0$  multiple of  $b - a$ . So I'm going to pull out this factor of  $b - a$ , and this row is going to become  $0, 1, b + a$ .

I will also, in the same step, do the same thing with the third row. I'll pull out a factor of  $c - a$ , and it will become  $0, 1, c + a$ . Here's one factor from the second row. Another factor from the third row. And  $1, a, a^2$ .  $0, 1, b + a$ .  $0, 1, c + a$ .

Now what? Well remember, you know how to do the determinant of upper triangular matrices because all that you do is multiply the pivots. This is almost upper triangular, except there's a  $1$  over here. So let's do another elimination step.  $b - a, c - a, 1, a, a^2$ .  $0, 1, b + a$ .  $0, 0, c + a - b + a$  is  $c - b$ .

So the determinant of this matrix is now  $1 \times 1 \times c - b$ . So we get  $b - a, c - a, c - b$ , which has a really nice formula. This is called the Vandermonde determinant. It's always like this, even if your matrix is bigger than  $3 \times 3$ , if it's  $4 \times 4$ , or  $5 \times 5$  and so on, you just have more differences of all the letters that show up in your matrix.

On to matrix number 3.  $C$  equals  $1, 2, 3$ .  $1, -4, 5$ . How did you get the determinant for this one? Well remember, this is a rank one matrix because it's a column vector times a row vector. So if you write out what the matrix is it'll be a  $3 \times 3$  matrix where we can think about it this way. The first row will be  $1 \times 1, -4, 5$ . The second row will be  $2$  times those numbers. And the third row will be  $3$  times those numbers.

So all the rows are going to be linearly dependent, or another way of saying the

matrix is singular. When the matrix is singular, the determinant is always equal to 0. That's also one of the properties.  $C$  is 0.

Onto the next matrix. Last one.  $D$  is equal to 0, 0, 0. 1, -1, 3, -3, 4, -4. Maybe you didn't get that from just looking at the matrix the first time, but did you see how I wrote it down? It has 0's down the diagonal. And then for each entry, I have minus that entry. That means that this matrix is skew symmetric. What that means is if you do  $D$  transpose, it will not be equal to  $D$ , but it will be equal to minus  $D$ .  $D$  transpose is equal to minus  $D$ .

Now, what does this give us for a determinant? Well if these two matrices are the same matrix, this determinant is equal to that determinant. One of the properties is that the determinant of a transpose of the matrix is equal to the determinant of the original matrix.

How about this side? Well first temptation would be to just write that. Is that always true? No. The determinant is linear on each row separately. That means that you can't pull out the factor that is multiplying the matrix. You have to pull it out once for each row of the matrix.

So what I should have written was -1, that's my factor minus. How many rows do I have? One, two, three. Pull it out once for each row times the determinant of  $D$ . Well fortunately, -1 to the third is simply equal to -1. So here we go. It was correct, in fact.

We have determinant of  $D$  is equal to minus determinant of  $D$ . What is the only number that is equal to minus that number? 0. Determinant of  $D$  is equal to 0 again.

Let me ask you one last question. Is it true that all skew symmetric matrices have the determinant equal to 0? It was true for this one. Is it true in every case? Well the key factor here was that I had -1 to the third power and I got a minus sign here, determinant of  $D$  is equal to minus determinant of  $D$ .

What if this number had been an even number? Then I would just have the determinant of  $D$  is equal minus 1 to an even number,  $D$ . That's 1. So I would have

determinant of  $D$  is equal to determinant of  $D$ . There's nothing I can say about that number. It can be your favorite number, not necessarily 0.

All right. We're done for today. Thank you.