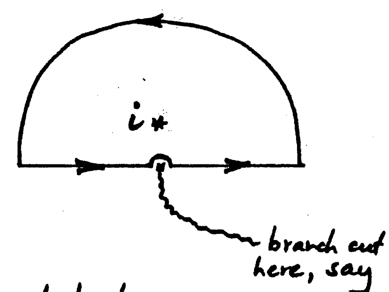
① Astonishingly, our monster $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$ = exactly **ZERO**!

One way to see this is by writing $u = \ln x$, or $x = e^u$. Then $dx = e^u du$, and this integral becomes

$$\int_{\boxed{u=-\infty}}^{\infty} \frac{u e^u}{1+e^{2u}}\,du = \int_{-\infty}^{\infty} \boxed{\frac{u}{e^u+e^{-u}}}\,du = 0$$

Careful with this lower limit! $u=0$ is a common mistake deserving not a great deal of sympathy...

Odd fctn. of u! Hence



Via residue calculus instead, it was important to use the path as hinted rather than the more customary "toilet seat" contour for which our desired real integral would unfortunately have been cancelled by the contribution from the return path immediately below the positive real axis (and also the cut).

Choosing the integrand $\frac{\log z}{z^2+1}$ to equal our given real function along the positive real axis, we would then find that it equals $\frac{\ln r + i\pi}{r^2+1}$ at the locations $z=-r$ along the negative real axis. More important still, that latter integral would begin at $r =$ huge R and end at $r =$ tiny ε, whereas neither the large semicircle nor the small one would contribute anything in the $R \to \infty$ and $\varepsilon \to 0+$ limits. All told, we would obtain

$$\int_{r=\infty}^{0} \frac{\ln r + i\pi}{r^2+1}\,\overset{=dx!}{(-dr)} + \int_{x=0}^{\infty} \frac{\ln x}{x^2+1}\,dx = 2\pi i\, \mathrm{Res}(i),$$

where

$$Res(i) \equiv \text{Residue of } \frac{\log z}{(z-i)(z+i)} \text{ at } z=i = \frac{i\pi/2}{i+i}.$$

Hence $$2 * \boxed{\text{our} \int} + i\pi \int_0^\infty \frac{dr}{r^2+1} = 2\pi i * \frac{\pi}{4} + \boxed{0}.$$

... since the REAL PARTS are these

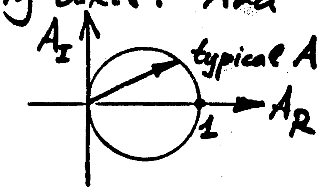
② a) Much more briefly, the mapping $W = \frac{z-1}{z+1}$ lands us within the unit circle $|W| = 1$ whenever the "distance" $|z-1|$ of point z from $+1$ is smaller than its "distance" $|z+1|$ from location -1; hence anywhere with $Re(z) > 0$ will do.

b) Similarly, the mapping $q = \frac{z-i}{z+i}$ will compress the entire upper half-plane into that unit circle, and $W(z) = q(z) + (1+i)$ will in turn slide that circle to the desired location. Cleaned up, $\boxed{W(z) = \frac{(2+i)z - 1}{z+i}}$.

③ The expression $x(t) = Re\{A(\omega)e^{i\omega t}\} = A_R \cos\omega t - A_I \sin\omega t$ is of course only an abbreviation for the cosine + sine(ωt) solutions which $dx/dt + x = \cos\omega t$ surely possesses. But via this complex route we save a lot of messy algebra.

Indeed pretend the "forcing" consists not of $\cos\omega t$ but of $e^{i\omega t}$, of which the former is only the real part. In that case the guess $x(t) = Ae^{i\omega t}$ leads promptly to $(i\omega + 1)A(\omega) = 1$, or to $\rightarrow$ $\boxed{A(\omega) = \frac{1}{1+i\omega}}$

Think of this denominator as $z = 1 + i\omega$ & of its locus for various ω's as a straight line at unit distance from the imaginary axis. And surely the "reciprocal" of that line is a circle such as the one shown:



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