18.04 Ancient History #1

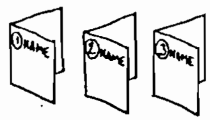
Fri 26 Sep 03

18.04 Exam #1

Friday, March 1, 1996

CLOSED BOOK

Again this year, please ...



1. Where in the complex z plane can we possibly be if told only that

$$\left|\frac{z-3}{z+3}\right| = 2 \quad ?$$

In other words, determine a more "civilized" name, shape and formula for this curious locus.

2. By means of a well-reasoned sketch, determine the net increase $2\pi N$ of the argument of the complex polynomial

$$f(z) = z^9 + 5z^2 + 1 = (z-z_1)(z-z_2)\ldots(z-z_9)$$

as one travels smoothly around the circle $|z| = 1$ from $z = 1$ back to $z = 1$ one full turn in the counterclockwise sense. Hence exactly how many of the nine zeroes $z_1, z_2, \ldots, z_9$ of the above polynomial must reside within the circle $|z| = 1$? Do explain your reasoning!

3. Locate all values of z for which $\cosh z = \sinh 2z$, or equivalently

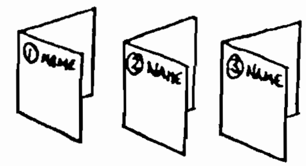
$$e^z + e^{-z} = e^{2z} - e^{-2z} \quad .$$

18.04 Exam #1

Friday, September 27, 1991

CLOSED BOOK

As is our custom, please struggle with each problem on a separate sheet of paper!



1. Apply your algebraic skills (rather than your calculator) to the product $(1+i)(5-i)^4$, and thereby rederive the awesome identity $\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$.

2. Likewise reaffirm that the function $w = z^2$ maps

   (a) the line $x + y = 1$, and

   (b) both branches of the hyperbola $x^2 = 1 + y^2$

   from the z-plane into a parabola and a single straight line, respectively, in the w-plane .

3. For any real θ evaluate the geometric sum

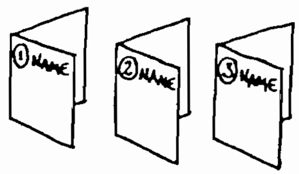
$$S(\theta) = \sum_{k=-5}^{5} e^{ik\theta}$$

as the ratio of two sines. HINT: Employ $e^{\pm i\theta/2}$.

18.04 Exam #1 Wednesday, February 27, 1991

CLOSED BOOK

As in past years, please struggle with Problems 1, 2 and 3 on separate sheets of paper ...



(1) With some eloquent words and sketches, identify clearly those regions of the complex z-plane for which

(a) $\mathrm{Re}\, z = \mathrm{Re}(z^3)$

(b) $z\bar{z} + z + \bar{z} < \mathrm{Im}\, z$

(2) Only one of the three functions

$$e^{-x}\cos(xy)\ , \quad x\sin y - y\sin x\ , \quad x^4 - 6x^2y^2 + y^4$$

can be the real part of some analytic function. Identify that valid candidate, and from it also deduce the imaginary part of this analytic function, insofar as possible.

(3) Locate all finite roots of

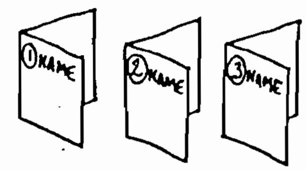
(a) $(1+z)^8 = (1-z)^8$

(b) $\cos z + \sin z = 0$

18.04 Exam #1 Friday, October 9, 1987

CLOSED BOOK

As in the past, please ...



(1) Whenever z_1, z_2, z_3 mark the vertices of an equilateral triangle in the complex z-plane, show that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \quad .$$

(2) Neatly and simply, where can we possibly be if:

(a) $z^6 + z^5 + z^4 + z^3 + z^2 = 0$

(b) $e^z + \cosh z = 0$

(3) For the non-analytic function $f(z) = x^2 + iy$, evaluate the integral

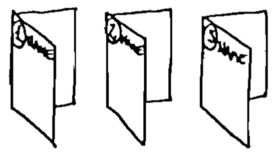
$$\oint f(z)\, dz$$

taken counterclockwise once around the unit circle $|z| = 1$.

18.04 Exam #1 Friday, March 7, 1997

CLOSED BOOK

Once again, please struggle with Problems 1, 2 and 3 on separate sheets of paper ...



and also indicate your RECITATION: M2 M3 Tu11 Tu1 Tu2 . Thanks!

(1) For any analytic function $f(z) = u + iv$, use the Cauchy-Riemann equations (which should be clearly stated but need not be rederived here) to show that the real function

$$P(x,y) = u(x,y)\, v(x,y)$$

that is the product of its real and imaginary parts must itself be harmonic — i.e., satisfy the Laplace equation.

(2) Show that the function $w = z^2$ maps

(a) the line $x = 1$,
(b) the hyperbola $xy = 1$, and
(c) the circle $|z-1| = 1$

from the z-plane respectively into a parabola, a straight line, and the cardioid $R = 2(1 + \cos\theta)$ in the w-plane.

(3) The inventor of the brilliant new function

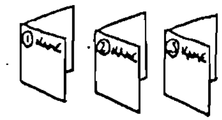
$$w = \text{itch}(z) = 2e^z + e^{2z}$$

wants to know its inverse $\text{itch}^{-1}(w)$ explicitly in terms of the complex logarithm. Please help ... and also display the power of your formula (or at least of your logic) by reporting all possible values of z for which $\text{itch}(z) = 3$.

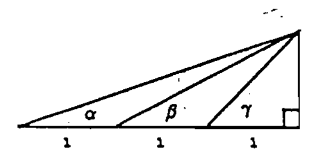
18.04 Exam #1 Friday, October 6, 2000

CLOSED BOOK ... and NO calculators

As before, please struggle with Problems 1, 2 and 3 on separate sheets of paper ...



(1) Use complex algebra to confirm that



(a) the sum $\alpha + \beta + \gamma$ of the three angles shown is $\pi/2$

(b) if $|z| = 1$, then $|z - w| = |1 - \bar{w}z|$ for any w

(2) Show that the function $w(z) = z + 1/z$ maps that "ray" or semi-infinite straight line for which $\arg(z) = \pi/3$ into a hyperbola in the $w = u + iv$ plane. Exactly what is the u,v equation for that hyperbola, what are its asymptotes, and where (if at all) does it cross the u or v axes?

(3) Consider that branch of the otherwise 4-valued function

$$f(z) = \sqrt{1 + \sqrt{z}}$$

for which $f(4) = +i$. Evaluate $f'(4)$ and also $f''(4)$, preferably by first rephrasing this problem as

$$f^2(z) = 1 + w(z) \quad \text{and} \quad w^2(z) = z \ ,$$

and then suitably differentiating those two formulas to cut through this mad confusion.

① Here we could start from the CR eqns $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ and simply "chug away":

$$\nabla^2(uv) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)uv = (u_{xx}+u_{yy})v + 2(u_x v_x + u_y v_y) + u(v_{xx}+v_{yy})$$

... where each of the three (...) can soon be shown to vanish.

But it is even smarter to think of the new analytic function $g(z) = \frac{1}{2}[f(z)]^2 = \frac{1}{2}(u^2-v^2) + iuv$, whose $Im(g) = uv$ itself!

② (a) $z = 1+ip \to w = (1+ip)^2 = (1-p^2) + 2ip$ = PARABOLA

(b) $z = p + \frac{i}{p} \to w = \left(p+\frac{i}{p}\right)^2 = \left(p^2 - \frac{1}{p^2}\right) + 2i$ = STR. LINE

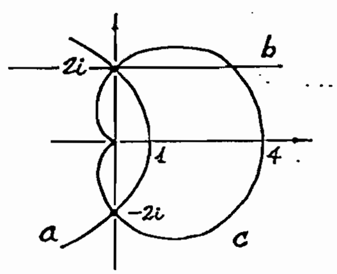
(c) $z = re^{i\alpha}$, with $r(\alpha) = 2\cos\alpha$

$\to w = Re^{i\theta}$ with $\theta = 2\alpha$

and $R = r^2 = 4\cos^2\alpha = 2 + 2\cos 2\alpha$

$= 2 + 2\cos\theta$ = CARDIOID

All three in a single w-plane picture:



③ From $w = 2e^z + e^{2z}$ it sure follows that $e^{2z} + 2e^z + 1 = w+1$, or that $(e^z+1)^2 = w+1$, does it not?

Hence $e^z + 1 = \sqrt{w+1}$ with the usual 2-fold ambiguity, and $e^z = \sqrt{w+1} - 1$, or $z = \log\left[\sqrt{w-1} - 1\right] = itch^{-1}(w)$

From the above, $w=3$ implies $e^z = \pm 2 - 1$, or z = either $2\pi i N$ or else $\ln 3 + \pi i + 2\pi i N$ for $N = 0, \pm 1, \pm 2$, etc.

AT

① a) Think of $(3+i)(2+i)(1+i) = (5+5i)(1+i) = 10i$, with $\measuredangle = 90°$

b) $|z-w| = |\overline{z-w}| = |\bar{z}-\bar{w}| = \left|\frac{1}{z} - \bar{w}\right| = \frac{|1-\bar{w}z|}{|z|} = |1-\bar{w}z|$

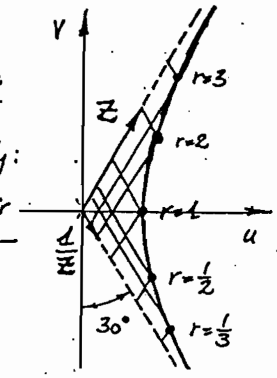
since $z\bar{z}=1$ here; likewise $=1$

② Writing $z = re^{i\pi/3} = r\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$, with "radius" r as our parameter, we have quickly that $1/z = \frac{1}{r}e^{-i\pi/3} = \frac{1}{r}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$ and thence that $w = u+iv = z + \frac{1}{z}$, where $u = \frac{1}{2}\left(r+\frac{1}{r}\right)$ and $v = \frac{\sqrt{3}}{2}\left(r - \frac{1}{r}\right)$. And this sure looks like a hyperbola, since here

$$\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4 = (2u)^2 - \frac{(2v)^2}{3}$$

or $u^2 - \frac{1}{3}v^2 = 1$. Or graphically:

Plainly this (half) of that hyperbola pair crosses only the u-axis — at u=1 — and its two asymptotes tilt ±30° from the "vertical" or v-axis.



③ With $f^2(z) = 1 + w(z)$ and $w^2(z) = z$ as suggested, we are evidently dealing not only with $f(4) = +i$ but also $w(4) = -2$.

Now differentiating, $2f\frac{df}{dz} = \frac{dw}{dz}$ ① and $2w\frac{dw}{dz} = 1$. ②

And using primes to abbreviate d/dz, we have also that

$2ff'' + 2(f')^2 = w''$ ③ and $2ww'' + 2(w')^2 = 0$. ④

The rest is just "plug-in". For instance, eqn. ② discloses at $z=4$ that $w'(4) = 1/2w = -1/4$, and eqn. ① follows with $f'(4) = \frac{w'}{2f} = \frac{i}{8}$. Similarly $w'' = \ldots$ and $f''(4) = -\frac{i}{32}$