Exam Number 2 for 18.04, MIT (Fall 1999).

Rodolfo R. Rosales* Boris Schlittgen† Zhaohui Zhang‡

Friday November 12, 1999 (room 6-120, 12:00 to 1:00 PM).

POINTS: all the problems are worth the same amount.

There is a total of FIVE problems.

1 Instructions.

1. **Follow this instructions. Penalties will result for failure to do so.**
2. **Exam is open book and open notes, to be done individually. You can consult with the exam proctor (and no one else) if you have a question.**
3. **Write your name clearly** (in print!) **on the exam.**
4. **Use clear and** $\boxed{\textbf{legible writing}}$**, with a pen/pencil dark enough. Answers in too tiny a font, or unreadable for any other reason will be considered as** $\boxed{\textbf{not there}}$. *In the last exam a magnifying glass (no kidding) and special lighting was needed to read some of the answers.* In this exam, answers with this sort of feature will get: $\boxed{\textbf{zero credit}}$.

   The same goes for half–erased–half–there–half–not–there answers. If something is part of the answer, it must be clearly there and if it is not it must be erased or crossed out completely. *Ambiguous answers* **will not** *be accepted.*
5. **Explain how you obtained your answers in** English **and mark where the answer is clearly and without ambiguity** .... $\boxed{\textbf{put a box around it!}}$

*MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

†MIT, Department of Mathematics, room 2-490, Cambridge, MA 02139.

‡MIT, Department of Mathematics, room 2-229, Cambridge, MA 02139.

Below some hints and tips on how to make things readable and on how to avoid being trapped into making preventable errors.

Notice that we will be sympathetic with algebraic and other simple errors that can be attributed to the rush and pressure of an exam, but we **will not be sympathetic with errors incurred because of sloppiness in the process of doing the problems!** There is a big difference between these two situations and the grading will reflect it.

- The *problems are doctored*, so that none of them requires heavy algebra. Read the statements carefully before you start; most of them require you to apply a particular concept or method introduced in the lectures or the problem sets. Once you identify it, the answer will follow in just a few lines.
- Bring scratch paper with you to the exam and do your scratch calculations there (**not** within your answers). The amount of such calculation should be minimal, at any rate, once you hit on the right way to do the problem. However, putting all your attempts into the exam will not only produce very hard to read answers, it will also significantly increase the risk of you ending up confused and with either a wrong or ambiguous answer in the exam!
- Do not use arrows leading from one piece of the argument to another part somewhere else and do not introduce needless ("funny") symbols into your answer, especially ones without a definition attached! Define what your variables are (in case you need to introduce new ones not in the problem statement). Again, failing to do these things not only produces very hard to read answers, but significantly increases the risk of ending up confused and with either a wrong or ambiguous answer in the exam! An error caused by (for example) having the same symbol meaning different things in different places (or not having a clearly defined meaning) is not a "small error" due to a forgivable confusion; it is an error due to sloppiness and as such, not very forgivable.
- *English means understanding.* Seriously; if you cannot explain your reasoning to some other (reasonably knowledgeable) person in simple terms, without need of tons of jargon *there is a pretty good chance that you do not understand the problem* in enough depth.

  It is a well known fact amongst researchers that, when you are stuck in a problem, it is often very helpful to talk with someone else. This is often the case even if the other person does not give you even one single useful hint during the conversation. The reason that this works is, mainly, that explaining the problem to someone else forces you to organize your thoughts in a clear manner not involving preconceived notions and unjustified assumptions. For this to work, all you need is a smart listener that will not buy something just because you say so and (furthermore) will not let you get away with jargon code words he or she does not understand.

2 Problems.

2.1 Problem 1999.2.1.

Statement:

Note: this is, we think, the longest (harder?) problem in the exam. You may want to postpone doing it till you have done the others.

Consider the branch $f = f(z)$ of the multiple valued function $\sqrt{16+z^2}$ given by:

- $f = f(z)$ is defined on the complex plane cut along the segment of the imaginary axis joining the two branch points $z = \pm 4i$.
- $f(3) = -5$.

Notice then that (on the real axis) $f = f(x) < 0$ for $x > 0$ and $f = f(x) > 0$ for $x < 0$ (in particular, $f(-3) = 5$). Consider now the three series:

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{n!} 2^n f^{(n)}(1), \qquad S_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 4^n f^{(n)}(1) \quad \text{and} \quad S_3 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 5^n f^{(n)}(1),$$

where $f^{(n)} = f^{(n)}(z)$ denotes the n^{th} derivative of f.

Questions: For $j = 1$, $j = 2$ and $j = 3$: does S_j converge? Why? If so, what is the value of of S_j?

Note: do not just write a value for S_j; explain how you obtained it!

Solution:

Continue answer to problem # 1:

2.2 Problem 1999.2.2.

Statement:

Consider the multiple valued function

$$f = f(z) = \log(z^2 + 1) + 3\log(z + 2) - a\log(z - 3)\,,$$

where **either** $a = 5$ **or** $a = 4$. Find all the branch points for this function.

Solution:

Continue answer to problem # 2:

2.3 Problem 1999.2.3.

Statement:

Consider the functions

$$f_1(z) = \cos(\sqrt{z})\,, \qquad f_2(z) = \sqrt{1+z^2}\,, \qquad f_3(z) = \cos(\sqrt{1+z^2}) \qquad \text{and} \qquad f_4(z) = \frac{1}{\sqrt{1+z^2}}\,.$$

Find their singular points in the finite complex plane and classify them (i.e.: isolated or non-isolated). Hint: *at least for f_1, think Taylor.*

Solution:

Continue answer to problem # 3:

2.4 Problem 1999.2.4.

Statement:

Let $T = T(x, y)$ be the (steady state) temperature distribution inside a flat disk of unit radius (i.e.: on $x^2 + y^2 < 1$). Assume that, **on the boundary of the disk** (which we parameterize by the angle θ, so that $x = \cos(\theta)$ and $y = \sin(\theta)$) **we have:**

$$T = \sin^2(\theta)\,.$$

Questions:

A) What is the temperature at the center of the disk?

B) What is the maximum value of the temperature in the disk $x^2 + y^2 \leq 1$?

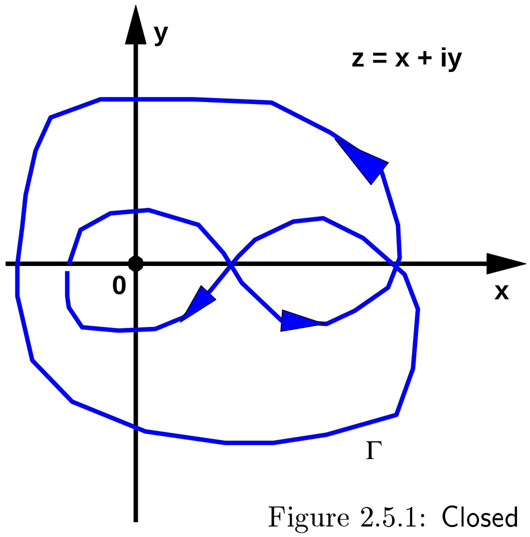
C) What is the minimum value of the temperature in the disk $x^2 + y^2 \leq 1$?

Solution:

Continue answer to problem # 4:

2.5 Problem 1999.2.5.

Statement:



For the closed contour Γ given in the figure on the left, calculate:

$$I = \oint_\Gamma \cos\left(\frac{1}{z}\right)\, dz\,.$$

Figure 2.5.1: Closed contour of integration Γ.

Solution:

Continue answer to problem # 5:

THE END.