Part II Problems and Solutions

Problem 1: *[Complex and repeated eigenvalues]*

(a) *The population of long-tailed weasels and meadow voles on Nantucket Island has been studied by biologists. They measure the populations relative to a baseline, in hundreds of animals. So $x(2) = 5$ means that at $t = 2$ there are 500 more weasels than the baseline value, and $y(2) = -5$ means that at $t = 2$ there are 500 fewer voles than the baseline value.*

Biologists from MIT have established the following relationship between $x(t)$ and $y(t)$:

$$\dot{x} = (0.5)x + y\,, \quad \dot{y} = -(2.25)x + (0.5)y$$

So the natural growth rate for each species is 0.5, and cross terms show that more voles are good for weasels but more weasels are very bad for voles.

Work out the population evolution predicted by this model. (`Linear Phase Portraits: Matrix Entry` *may be useful to you in visualizing this.) What are the eigenvalues? For one eigenvalue, find the exponential solution of $\dot{\mathbf{u}} = A\mathbf{u}$. Then take the real and imaginary parts to produce two real solutions. Suppose that at $t = 0$ there are 100 more weasels than baseline, and the vole population is the baseline value. What is the solution to this initial value problem? Is the number of weasels increasing at $t = 0$? How about the number of voles? When does the number of voles next equal its baseline value (if ever)? Please sketch the phase portrait and the graphs of $x(t)$ and $y(t)$ (for this initial condition) for t between about -2 and $+2$. Make it clear how these three pictures are related: indicate on the trajectory the position of $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ when $t = \frac{k\pi}{3}$ for $k = -2, -1, 0, 1, 2$.*

(b) *Invoke the Mathlet* `Linear Phase Portraits: Matrix Entry`*. Play with the tool for a while to get a feel of it. Notice that the eigenvalues can be displayed on the complex plane. Deselect the* `[Companion Matrix]` *option, so you can set all four entries in the matrix. Select the* `[eigenvalues]` *option, so the eigenvalues become visible by means of a plot of their location in the complex plane and also a read-out of their values.*

Set the sliders so that $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Watch what happens to the phase portrait as b varies from -4 to 4. For each value of b, the applet suggests a pair of eigenvalues; verify what it claims. For each value of b, describe the collection of all eigenvectors for each eigenvalue. (This description will depend upon b.) For each value of b, write down all the normal mode solutions (i.e. all the solutions of the form $e^{\lambda t}\mathbf{v}$, where λ is a number and $\mathbf{v}$ is a vector). If the system is defective, find a complementary solution. Take one value of b for which the system is complete and sketch the phase portrait. Take one value of b for which the system is defective and sketch the phase portrait.

Solution: **(a)** $A = \begin{bmatrix} 0.5 & 1 \\ -2.25 & 0.5 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - \lambda + 2.5$, and eigenvalues $\frac{1 \pm 3i}{2}$. An eigenvector for $\lambda_1 = \frac{1+3i}{2}$ satisfies $(A - \lambda_1 I)\mathbf{v_1} = \mathbf{0}$, that is, $\begin{bmatrix} -3i/2 & 1 \\ -9/4 & -3i/2 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$. One choice is $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3i/2 \end{bmatrix}$. The normal mode is then $e^{(1+3i)t/2} \begin{bmatrix} 1 \\ 3i/2 \end{bmatrix}$, which has real and imaginary parts $\mathbf{u_1} = e^{t/2} \begin{bmatrix} \cos(3t/2) \\ -(3/2)\sin(3t/2) \end{bmatrix}$ and $\mathbf{u_2} = e^{t/2} \begin{bmatrix} \sin(3t/2) \\ (3/2)\cos(3t/2) \end{bmatrix}$.

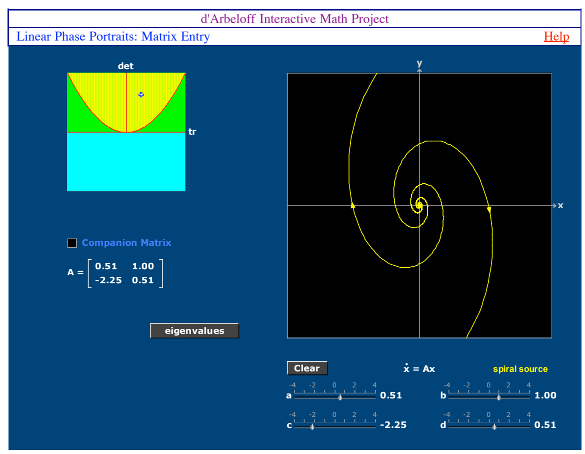
The initial condition is $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which, conveniently, is satisfied by $\mathbf{u_1}$. Since $\dot{\mathbf{u}} = A\mathbf{u}$, we find $\dot{\mathbf{u}}(0) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2.25 \end{bmatrix}$. So $\dot{x}(0) = 0.5$, and the weasel population is increasing at $t = 0$ while the number of voles is decreasing. $y(t) = 0$ occurs next when $3t/2 = \pi$, or $t = 2\pi/3$.

The graphs of $x(t) = e^{t/2}\cos(3t/2)$ and $y(t) = -(3/2)e^{t/2}\sin(3t/2)$ are "anti-damped" sinusoids, with increasing amplitude. The relevant trajectory is the one crossing the positive x axis half way out. The values of $\mathbf{u}(t)$ are

$$\mathbf{u}\left(-\tfrac{2\pi}{3}\right) = e^{-\pi/3} \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{u}\left(-\tfrac{\pi}{3}\right) = e^{-\pi/6} \begin{bmatrix} 0 \\ 3/2 \end{bmatrix},$$

$$\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u}\left(\tfrac{\pi}{3}\right) = e^{\pi/6} \begin{bmatrix} 0 \\ -3/2 \end{bmatrix},$$

$$\mathbf{u}\left(\tfrac{2\pi}{3}\right) = e^{\pi/3} \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$



(b) [8] With $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $p_A(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so we have a repeated eigenvalue $\lambda_1 = 1$. To find an eigenvector form $A - \lambda_1 I = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. A nonzero eigenvector is given (for any b) by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $b \neq 0$, the eigenvectors for value λ_1 are exactly the multiples of $\mathbf{v}$ (the matrix is defective), but for $b = 0$, $A = I$ and any vector is an eigenvector (the matrix is complete). When $b \neq 0$, the normal modes are $e^t \begin{bmatrix} c \\ 0 \end{bmatrix}$, for c a real constant. When $b = 0$, the normal modes are $e^t\mathbf{v}$ for any vector $\mathbf{v}$. When $b \neq 0$, we must solve $(A - \lambda_1 I)\mathbf{w} = \mathbf{v_1}$, that is, $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The solution is $\mathbf{w} = \begin{bmatrix} 0 \\ 1/b \end{bmatrix}$, so the extra solution is $\mathbf{u_2} = e^{\lambda_1 t}(t\mathbf{v_1} + \mathbf{w}) = e^t \begin{bmatrix} t \\ 1/b \end{bmatrix}$.

Problem 2: *[Qualitative behavior of linear systems]*

We will use `Linear Phase Portraits: Matrix Entry` *to investigate the phase portraits of the homogeneous linear equation $\dot{\mathbf{u}} = A\mathbf{u}$, where $A = \begin{bmatrix} a & -3 \\ 1 & -1 \end{bmatrix}$, as a varies. The focus is now on the colorful diagram at the upper left. To start with, set the matrix to $A = \begin{bmatrix} -4 & -3 \\ 1 & -1 \end{bmatrix}$. Then move the a slider up to $a = 4$, and watch (1) the movement of the mark on the (Tr,Det) plane; (2) the movement of the eigenvalues in the complex plane; and (3) the variation of the phase portrait.*

(a) *Compute the trace and determinant of A. (They will depend upon a, of course.) Find an equation for the curve (or line) traced out by the mark on the (Tr,Det) plane.*

(b) *You notice that the curve in the (Tr,Det) plane enters a number of different regions. When the cursor crosses a red boundary, the trajectories and the eigenvalue indicators turn red. Work out what the values of a are at those crossings. (So this is: where $\det A = 0$, where $\det A = (\mathrm{tr}A/2)^2$ (twice, once not represented on the Mathlet), and where $\mathrm{tr}A = 0$.*

(c) *There are nine phase portrait types represented as a varies (five regions and four walls). Draw an interval from -5 to $+4$. On it, mark the four values of a at which the matrix crosses one of the walls. Indicate the type of phase portrait you have at each of the marked points and along the intervals between them. That is, classify the phase portrait into one of the following types, as in the Supplementary Notes, §25: spiral (stable/unstable, clockwise/counterclockwise), node (stable/unstable); saddle; center (clockwise/counterclockwise); star (stable/unstable); defective node (stable/unstable; clockwise/counterclockwise); degenerate (comb (stable/unstable), constant, parallel lines). (The applet uses the alternative terms "source" and "sink" for "unstable" and "stable.")*

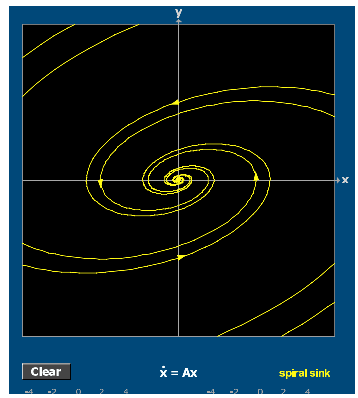
(d) *For each of the four special values, and for your choice of one value in each of the five regons, make a sketch of the phase portrait. Be sure to include and mark as such any eigenlines, and the direction of time.*

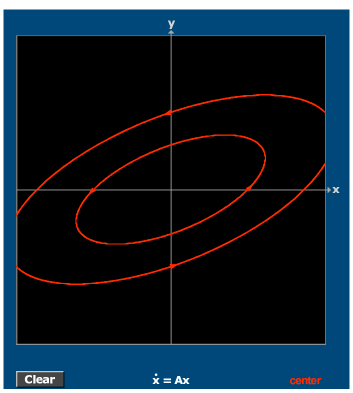
Solution: **(a)** $\mathrm{tr}A = a - 1$, $\det A = 3 - a$, so $\mathrm{tr}A = 2 - \det A$. $\det A = 0$ when $a = 3$. $\mathrm{tr}A = 0$ when $a = 1$. $\det A = (\mathrm{tr}A)^2/4$ when $a^2 + 2a - 11 = 0$ or $a = -1 \pm 2\sqrt{3}$, i.e. $a \simeq -4.4641$ and $a = 2.4641$.

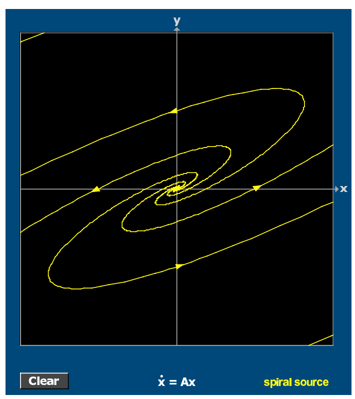
(c) Diagram showing: $a < -1 - 2\sqrt{3}$—stable node = nodal sink
$a = -1 - 2\sqrt{3}$—defective stable node = defective nodal sink

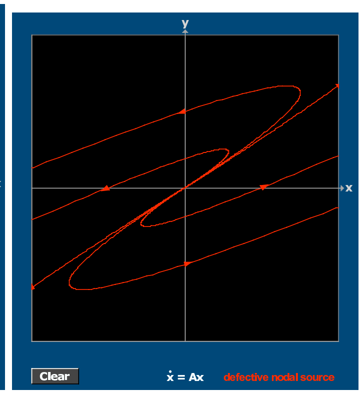
$-1-2\sqrt{3} < a < 1$—counterclockwise stable spiral = spiral sink
$a = 1$—counterclockwise center
$1 < a < -1+2\sqrt{3}$–counterclockwise unstable spiral = spiral source
$a = 1+2\sqrt{3}$—unstable defective node = defective nodal source
$1+2\sqrt{3} < a < 3$—unstable node = nodal source
$a = 3$—unstable degenerate comb
$3 < a$—saddle

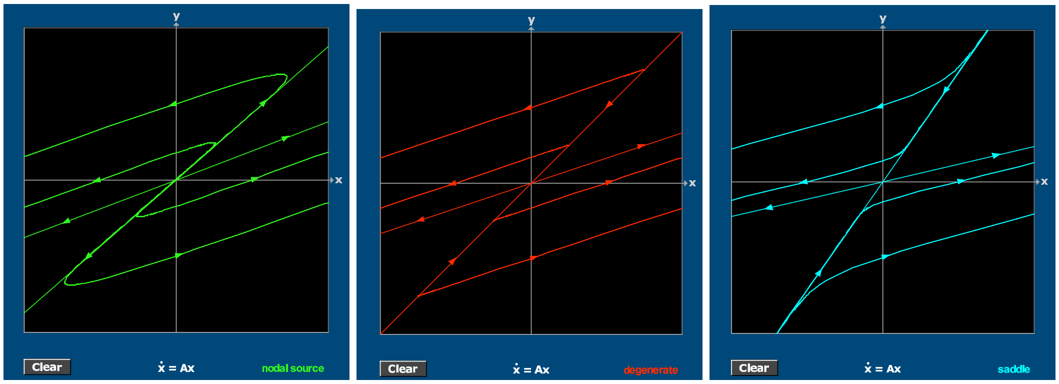
(b)-(c) Here are pictures for $a = 0, 1, 2, -1+2\sqrt{3}, 2.75, 3, 4$. ($a = -2\sqrt{3}$ omitted.) The picture for some $a < -1-2\sqrt{3}$ would show a nodal sink, and that for $a = -1-2\sqrt{3}$ would show a defective nodal sink.











Problem 3: *Let c be a real constant. This problem will analyze the system*

$$\begin{aligned} x' &= y \\ y' &= cx - 2y, \end{aligned}$$

(a) *What is the characteristic polynomial of the coefficient matrix A for the system?*

(b) *Compute its eigenvalues and eigenvectors. The answer depends on c so you will need to break your answer into cases.*

(c) *Write down the general* real *solution to the equation. Again you will need to break into cases depending on c. (The trace-determinant diagram will help.)*

(d) *Open the visual Linear Phase Portraits: Matrix Entry, click the eigenvalues button on and the companion matrix button off. Using representative values of c give sketches of all the different types of phase portraits possible as c varies. Using your answer in part (c) explain the portrait when $c = -3$.*

Solution:

(a) $p(\lambda) = \lambda^2 + 2\lambda - c$.

(b) Eigenvalues: $\lambda_1, \lambda_2 = -1 \pm \sqrt{1+c}$.

We can write the eigenvectors with just two cases.

i) $c \neq -1$: two different eigenvalues, eigenvector corresponding to λ_j is $\mathbf{v} = \begin{pmatrix} 1 \\ \lambda_j \end{pmatrix}$.

ii) $c = -1$: repeated eigenvalue, $\lambda = -1$

eigenvector = $\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, generalized eigenvector = $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(Recall, this means $A\mathbf{v_2} - \lambda\mathbf{v_2} = \mathbf{v_1}$. The vector $\mathbf{v_2}$ is not unique.)

(c) We determined all possible cases using the trace-determinant diagram below.

i) $-1 < c$: real eigenvalues, $\lambda_1, \lambda_2 = -1 \pm \sqrt{1+c}$.

Using part (b), general real solution: $\boxed{\mathbf{x} = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.}$

Subcases:

i.1) $0 < c$: 1 negative, one positive eigenval.: saddle (unstable).

i.2) $c = 0$: eigenvalues are negative and 0: line of critical points.

i.3) $-1 < c < 0$: 2 unequal negative eigenvalues: nodal sink (asymp. stable).

ii) $c = -1$: 1 eigenval. = -1, one eigenvector: improper nodal sink

General sol.: $\boxed{\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\left(t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).}$

iii) $c < -1$: complex roots with negative real part: spiral sink (asymp. stable)

Let $\omega = \sqrt{-c-1}$. This has eigenvalue/vector $\lambda = -1 + i\omega$, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 + i\omega \end{pmatrix}$. (and the complex conjugates).

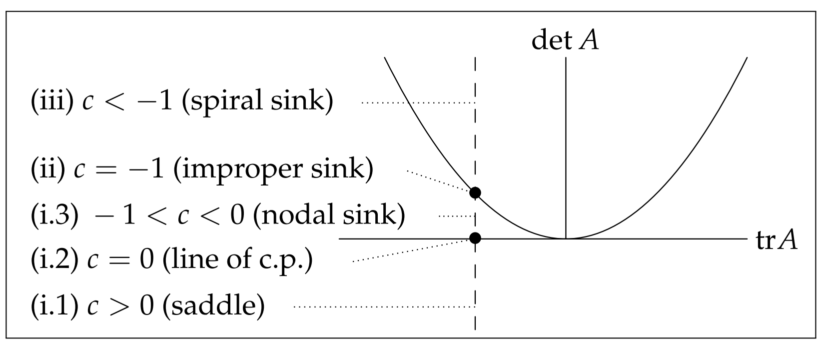
One complex solution: $\tilde{\mathbf{x}}_1 = e^{\lambda t}\mathbf{v} = e^{-t}\begin{pmatrix} \cos\omega t + i\,\sin\omega t \\ (-\cos\omega t - \omega\sin\omega t) + i\,(-\sin\omega t + \omega\cos\omega t) \end{pmatrix}$

general real solution: $\boxed{\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos\omega t \\ -\cos\omega t - \omega\sin\omega t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin\omega t \\ -\sin\omega t + \omega\cos\omega t \end{pmatrix}.}$

Coeff. matrix $A = \begin{pmatrix} 0 & 1 \\ c & -2 \end{pmatrix}$

$\Rightarrow$ tr$A = -2$ and $\det A = -c$.

In the trace-det. plane A is on the vertical line tr$A = -2$. This is the dashed line in the picture. It passes through 5 cases as shown.



(d) We match the diagrams to the classification in part (c) as follows:

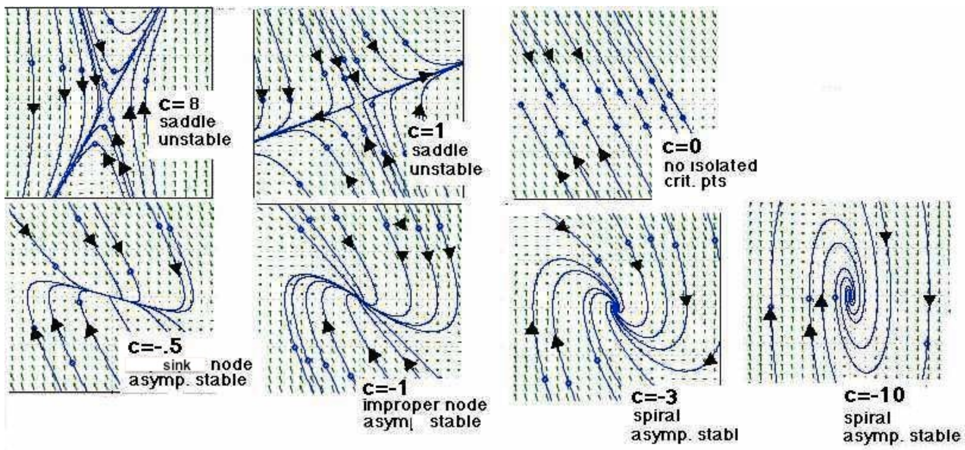
(i.1) $c = 8$, $c = 1$,

(i.2) $c = 0$,

(i.3) $c = -.5$,

(ii) $c = -1$,

(iii) $c = -2$, $c = -10$.



When $c = -3$ the portrait is a spiral towards the origin. The answer to part (c) says that the eigenvalues are complex with negative real part. The imaginary part leads to sin and cos in the solution, which causes the circular part of the spiral. The negative real part makes it spiral in to the origin as $t \to \infty$.