## Existence and Uniqueness and Superposition in the General Case

We can extend the results above to the inhomogeneous case.

$$
\begin{align*}
& \mathbf{x}^{\prime}=A(t) \mathbf{x} \text { (homogeneous) }  \tag{H}\\
& \mathbf{x}^{\prime}=A(t) \mathbf{x}+\mathbf{F}(t) \text { (inhomogeneous), } \tag{I}
\end{align*}
$$

where $F(t)$ is the input to the system.

## Linearity/superposition:

1. If $\mathbf{x}_{\mathbf{1}}$ and $\mathbf{x}_{\mathbf{2}}$ are solutions to (H) then so is $\mathbf{x}=c_{1} \mathbf{x}_{\mathbf{1}}+c_{2} \mathbf{x}_{\mathbf{2}}$
2. If $\mathbf{x}_{\mathbf{h}}$ is a solution to (H) and $\mathbf{x}_{\mathbf{p}}$ is a solution to (I) then $\mathbf{x}=\mathbf{x}_{\mathbf{h}}+\mathbf{x}_{\mathbf{p}}$ is also a solution to (I).
3. If $\mathbf{x}_{\mathbf{1}}{ }^{\prime}=A \mathbf{x}_{\mathbf{1}}+\mathbf{F}_{\mathbf{1}}$, and $\mathbf{x}_{\mathbf{2}}{ }^{\prime}=A \mathbf{x}_{\mathbf{2}}+\mathbf{F}_{\mathbf{2}}$ then $\mathbf{x}=\mathbf{x}_{\mathbf{1}}+\mathbf{x}_{\mathbf{2}}$ satisfies $\mathbf{x}^{\prime}=A \mathbf{x}+\mathbf{F}_{\mathbf{1}}+\mathbf{F}_{\mathbf{2}}$. That is, superposition of inputs leads to superposition of outputs.
proof: 1. $\mathbf{x}^{\prime}=c_{1} \mathbf{x}_{\mathbf{1}}{ }^{\prime}+c_{2} \mathbf{x}_{\mathbf{2}}{ }^{\prime}=c_{1} A \mathbf{x}_{\mathbf{1}}+c_{2} A \mathbf{x}_{\mathbf{2}}=A\left(c_{1} \mathbf{x}_{\mathbf{1}}+c_{2} \mathbf{x}_{\mathbf{2}}\right)=A \mathbf{x}$.
4. $\mathbf{x}^{\prime}=\mathbf{x}_{\mathbf{h}}{ }^{\prime}+\mathbf{x}_{\mathbf{p}}{ }^{\prime}=A \mathbf{x}_{\mathbf{h}}+A \mathbf{x}_{\mathbf{p}}+\mathbf{F}=A\left(\mathbf{x}_{\mathbf{h}}+\mathbf{x}_{\mathbf{p}}\right)+\mathbf{F}=A \mathbf{x}+\mathbf{F}$.
5. $\mathbf{x}^{\prime}=\mathbf{x}_{1}{ }^{\prime}+\mathbf{x}_{\mathbf{2}}{ }^{\prime}=A \mathbf{x}_{\mathbf{1}}+\mathbf{F}_{1}+A \mathbf{x}_{\mathbf{2}}+\mathbf{F}_{\mathbf{2}}=A\left(\mathbf{x}_{\mathbf{1}}+\mathbf{x}_{\mathbf{2}}\right)+\mathbf{F}_{\mathbf{1}}+\mathbf{F}_{\mathbf{2}}=$ $A \mathbf{x}+\mathbf{F}_{\mathbf{1}}+\mathbf{F}_{\mathbf{2}}$.

Existence and uniqueness: We start with an initial time $t_{0}$ and the initial value problem:

$$
\begin{equation*}
\mathbf{x}^{\prime}=A(t) \mathbf{x}+\mathbf{F}(t), \mathbf{x}\left(t_{0}\right)=\mathbf{x}_{\mathbf{0}} \tag{IVP}
\end{equation*}
$$

Theorem: If $A(t)$ and $\mathbf{F}(t)$ are continuous then there exists a unique solution to (IVP).

MIT OpenCourseWare
http://ocw.mit.edu

### 18.03SC Differential Equations

Fall 2011

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.

