

**PROFESSOR:** Hi, everyone. Welcome back. In this problem, I'd like to take a look at convolution and Green's formula. Specifically, for part one, we're just asked to compute the convolution of  $t$  with itself,  $t$ . In the second part, we're asked to compute the convolution of  $e^{-kt}$  with another exponential,  $e^{at}$ . And then in part B, we're to use the result from part A as well as Green's formula to solve  $x' + kx = e^{at}$ . And specifically, we're interested in the rest initial condition solution to this formula. OK. So I'll let you think about this problem, and I'll be back in a minute.

Hi, everyone. Welcome back. OK, so for part one, we're asked to compute the convolution of two functions,  $t$  with itself, and I'll just quickly write down the formula for a convolution. So if I have two functions,  $f$  and  $g$ , then the convolution is defined as, in this case, the integral from zero to  $t$  of  $f$ , so the function  $f$ , evaluated at, we're going to use  $\tau$  as the integration variable, multiply by  $g(t - \tau) d\tau$ . So notice how the variable  $t$  appears in two places in this formula. It appears in one place in the argument of  $g$ , and it appears in the bound of integration. Meanwhile,  $\tau$  is the variable we're integrating over, and  $\tau$  appears in two places as well, one in  $f$  and one in  $g$ .

So if we're interested in  $t$  convolved with  $t$ , we have the integral from zero to  $t$  of  $\tau t d\tau$ . So notice how  $f$  is just  $t$  and  $g$  is  $t$ , so when we insert  $\tau$  and  $t - \tau$  into the function, we end up just getting  $\tau t d\tau$ . And just expanding this out, we have  $t \tau^2 d\tau$ . So in the integral, the variable  $t$  always appears just as a constant. So in this case, we have  $t$ ,  $\tau$  integrated becomes  $\tau^2 / 2$ . When I integrate from zero to  $t$ ,  $\tau^2$  integrates to  $1/3 t^3$ , and zero drops out from the other end of the integral. So we end up with  $t^3 / 3 - 0$ , which is equal to  $t^3 / 6$ . So there's the answer we're looking for.

For part B, or for question two, again we have another computation to do. So we have more integrals to work out. And in this case, it's two exponentials. So we have zero to  $t$ . The first one is going to be  $e^{kt}$ . And the second one is  $e^{at}$ . So we can just expand things out, and I get  $e^{kt} \int e^{at} dt$ , zero to  $t$ . And again, because  $t$  is just a dummy variable in this integral, we're not integrating over  $t$ , I can just think of  $e^{at}$  as a constant. So I'm really just integrating the function  $e^{at}$  minus  $a$  plus  $k\tau$ . And when I integrate this function, I get  $1/(a+k)$ ,  $e^{at} - a$  plus  $k\tau$ .

minus a plus k, tau evaluated at the bounds, zero and t. So the negative sign here just flips the bounds. So we have a plus k. Substituting in zero just gives us 1, and then substituting in t gives us  $e^{-a} + e^{-kt}$ . And I can expand out, multiply out,  $e^{-a} + e^{-at}$ . And when the dust settles, we have  $e^{-at} - e^{-kt}$ .

So this is just the computation of the convolution  $e^{-a} * e^{-kt}$ . So note now if, for example, k were to equal negative a, we would have zero in the denominator and zero on the numerator. So one way to compute the special case when k is equal to negative a is to use L'Hopital's Rule and differentiate the top and the bottom, for example, with respect to k. Another alternative way of computing the special case when a is equal to negative k is just to plug in  $e^{-a}$ , and then just work out this integral, and you'll obtain a different answer.

And now lastly, for part B, we're interested in the differential equation  $x' + kx = e^{-at}$ . And we're interested in finding the particular solution that has rest initial conditions. So for example,  $x(0) = 0$ . And we want to use Green's formula. So just recall that there's the impulse response formula for the weight function w, which solves the same differential equation, but with the delta function on the right-hand side. And w, in this case, as we've seen in lecture, is actually  $e^{-kt}$  when  $t > 0$ , and at zero when  $t < 0$ .

So notice how  $e^{-kt}$  is exactly the function that we convolved in part A. Specifically, we convolved it with  $e^{-at}$ . So for example, Green's formula says that the particular solution that has rest initial conditions is going to be the weight function convolved with the right-hand side of the differential equation,  $e^{-at}$ . In this case, on the domain of integration for the convolution, w is just  $e^{-kt}$ . So we have  $e^{-kt}$  convolved with  $e^{-at}$ .

So Green's formula gives us the solution to this differential equation, which has rest initial conditions. And specifically-- we've already worked this out from part A, so I can just write down what the answer is. It's  $\frac{1}{a+k} e^{-at} - \frac{1}{a+k} e^{-kt}$ . And just as a quick check to make sure that we've done everything correctly, we can plug in  $t = 0$ . And when  $t = 0$ , we have  $1 - 1 = 0$ , which gives us zero. So indeed, this  $x$  is the solution to this differential equation, which has rest initial conditions.

I'd just like to conclude this problem. Quick recap. We worked out several convolutions, and specifically we were able to use Green's formula in addition to the convolution that we worked

out to compute the solution with rest initial conditions to an ODE. Moreover, Green's function is very useful because notice how we could have computed the convolution for any right-hand side function here. So in some sense, we're able to generalize and find solutions to differential equations which have arbitrary right-hand side forcings, not just sines and cosines, periodic functions, or other simple functions which we've been looking at in the past.

So I'd like to conclude here, and I'll see you next time.