18.034 PROBLEM SET 1

Due date: Friday, February 13 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Problem 1(20 points) The logistic model for a fish population with harvesting (p. 17) leads to the following IVP:

$$\begin{cases} y' = ay - cy^2 - H, \\ y(0) = y_0 \end{cases}$$

Here a and y_0 are positive and c and H are nonnegative. The IVP is defined on the interval $(0, \infty)$. Also, the model is only valid as long as $y(t) \geq 0$: If at any instant t_1 (greater than 0) $y(t_1)$ equals 0, then the population is extinct, and the population will remain extinct for all $t \geq t_1$.

(a)(10 points) The *equilibrium solutions* are the solutions of the ODE (without the initial condition) for which $y'(t) = 0$ for all t. Find inequalities among a, c, and H that determine when there will be 2 equilibrium solutions, 1 equilibrium solution, or no equilibrium solutions.

(b)(10 points) Suppose that both a and c are positive. What is the maximum value of H for which there is an equilibrium solution? If H is larger than this value, what is the long-term behavior of any solution of the ODE?

Problem 2(20 points) After a change of variables, the logistic equation with harvesting reduces to the following IVP (neglecting the extinction issue),

$$\begin{cases} x' = -x^2 + K, \\ x(0) = x_0 > 0 \end{cases}$$

where $x = x(t)$ and where K is a constant. Suppose that $K = b^2$ for some $b > 0$.

(a)(10 points) Formally rewrite the ODE as $f(x)dx = g(t)dt$ and integrate to find an exact solution. Express your answer in the form $b - x = h(t)$ for some expresion $h(t)$. Don't forget the special case $x_0 = b$.

(b)(10 points) At some instant t_1, the value of $x(t_1)$ is very close to b. At that instant, the value of b in the differential equation is abruptly increased to a larger value b_1, and $x(t)$ gradually moves from the value b to the value b_1. Assuming $b_1 - b$ is small compared to b, approximately how much time τ elapses before the difference $b_1 - x(t_1 + \tau)$ is one half of the initial difference $b_1 - b$?

(c)(0 points – not to be written up/handed in). Critical ecosystem double whammy. Interpret your answer from (b). In particular, if the parameters a, c and H are near the critical value for extinction, does the system respond more quickly or less quickly to a decrease in H than if the parameters are far from the critical value?

Problem 3(5 points) Exercise 14, p. 49.

Problem 4(5 points) Exercise 20, p. 49.