18.03 Recitation 16, April 6, 2010

Step and delta functions, and step and delta responses

1. Let $Q(t) = \begin{cases} 0 & \text{for } t < 1 \\ 2t-2 & \text{for } 1 < t < 2 \\ 2t-1 & \text{for } 2 < t < 3 \\ 5 & \text{for } 3 < t \end{cases}$

(a) Sketch a graph of this function. Is it piecewise smooth?

(b) Find the generalized derivative $q(t) = Q'(t)$, and sketch it.

(c) Tell a story which might result in the equation $\dot{x} + kx = q(t)$ with rest initial conditions. (Your choice of k, it might be negative).

(d) Tell a story which might result in the equation $2\ddot{x} + 4\dot{x} + 4x = q(t)$ with rest initial conditions.

$Q(t)$ is piecewise smooth, and its generalized derivative is $q(t) = 2u(t-1) + \delta(t-2) - 2u(t-3)$. The graph of $q(t)$ has a horizontal line $q = 0$ away from the interval $[1, 3]$. Inside $(1, 3)$, the graph takes the value 2, except at $t = 2$, where we have a harpoon labeled by 1.

Typical story for $\dot{x} + kx = q(t)$: x describes the balance of a bank account which grows through interest at rate k, and q represents the rate of savings. Before time 1 the account balance is zero. Between time 1 and time 3, the owner of the account has a job and steadily puts yearly 2 units of wealth into the bank account. At time 2, the owner wins a lottery, and deposits one unit of wealth.

Typical story for $2\ddot{x} + 4\dot{x} + 4x = q(t)$: $2\ddot{x} + 4\dot{x} + 4x$ describes a spring-mass-dashpot system (m=2, b=4, k=4) driven directly by the external force $q(t)$. Before time 1, the force is zero, the spring and the dashpot are relaxed and the mass is still. Between time 1 and time 3, the force is steadily 2 units. At time 2, a shock of one unit hits the system through the force.

2. Find the unit step and unit impulse responses for the operator $2D + I$, and graph them.

The unit step response is the continuous solution that is zero for $t < 0$, a solultion of $2\dot{x} + x = 1$ for $t > 0$. The general solution to the equation for $t > 0$ can be found by adding the general homogeneous solution to the particular solution $x_p = 1$. $2\dot{x} + x = 0$ has the general solution $ce^{-\frac{t}{2}}$, so the general solution of $2\dot{x} + x = 1$ is $x = 1 + ce^{-\frac{t}{2}}$. By the continuity condition at zero, x approaches zero as t approaches zero from above, so $c = -1$. Thus, the unit step response is $x = 1 - e^{-\frac{t}{2}}$ for $t \geq 0$ and $x = 0$ for $t < 0$.

The unit impulse response is the solution that is zero for $x < 0$, a solution of $2\dot{x} + x = 0$ for $x > 0$, and with x approaching $1/2$ as t approaches zero from above. Or equivalently, the unit impulse response is the derivative of the unit step response, so $x = \frac{1}{2}e^{-\frac{t}{2}}$ for $t > 0$ and $x = 0$ for $t < 0$.

3. Find the unit impulse response for the operator $D^2 + 2D$, and graph it.

We first find the unit step response for that operator. The unit step response is the continous solution that is zero for $x < 0$, a solution to the equation $\ddot{x} + 2\dot{x} = 1$ for $x > 0$, and with continuous first derivative at zero. The particular solution to the equation for $t > 0$ can be found by undetermined coefficient and it's $x_p = t/2$. The homogeneous solutions have the form $c_1 e^{-2t} + c_2$. So the general solution to $\ddot{x} + 2\dot{x} = 1$ is $x = t/2 + c_1 e^{-2t} + c_2$. By continuity at zero, $c_1 + c_2 = 0$, and by continuity of the first derivative $1/2 - 2c_1 = 0$, so $c_1 = 1/4$, $c_2 = -1/4$. Thus, the unit step response is $x = t/2 + (e^{-2t} - 1)/4$ for $t \geq 0$ and $x = 0$ for $t < 0$.

The unit impulse response is the derivative of the unit step response, so it's $x = (1 - e^{-2t})/2$ for $t \geq 0$ and $x = 0$ for $t < 0$.

4. From your answer to **3.**, find the solution to $\ddot{x} + 2\dot{x} = 3\delta(t - 1)$ with rest initial conditions.

Using time invariance, we find that a solution to $\ddot{x} + 2\dot{x} = 3\delta(t-1)$ is $x = \frac{3}{2}(1 - e^2 e^{-2t})$ when $t \geq 1$ and $x = 0$ when $0 \leq t < 1$, and it satisifes the rest initial conditions.

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