Recitation 3, February 9, 2010

Euler's method; Linear models

1. Use Euler's method to estimate the value at $x = 1.5$ of the solution of $y' = y^2 - x^2 = F(x, y)$ at $y(0) = -1$. Use $h = 0.5$. Recall the notation $x_0 = 0$, $y_0 = -1$, $x_{n+1} = x_n + h$, $y_{n+1} = y_n + m_n h$, $m_n = F(x_n, y_n)$. Make a table with columns n, x_n, y_n, m_n, $m_n h$. Draw the Euler polygon.

2. Is the estimate from **1.** likely to be too large or too small?

3. Here's a "mixing problem." A tank holds V liters of salt water. Suppose that a saline solution with concentration of c gm/liter is added at the rate of r liters/minute. A mixer keeps the salt essentially uniformly distributed in the tank. A pipe lets solution out of the tank at the same rate of r liters/minute. Write down the differential equation for the *amount* of salt in the tank. [Not the concentration!] Use the notation $x(t)$ for the number of grams of salt in the tank at time t. Check the units in your equation! Write it in standard linear form.

4. Now assume that c and r are constant; in fact, assume that $V = 1$ and $r = 2$. Solve this equation, under the assumption that $x(0) = 0$.

What is the limiting amount of salt in the tank? Does your result jibe with simple logic? When will the tank contain half that amount?

5. Now suppose that the out-flow from this tank leads into another tank, also of volume 1, and that at time $t = 1$ the water in it has no salt in it. Again there is a mixer and an outflow. Write down a differential equation for the amount of salt in this second tank, as a function of time.

6. Draw a picture of the circuit with a voltage source, a resistor, and a capacitor, in series. Denote by $I(t)$ the current (where the positive direction is, say, clockwise) in the circuit and by $V(t)$ the voltage increase across the voltage source, at time t. Denote by R the resistance of the resistor and C the capacitance of the capacitor (in units which we will not specify)—both positive numbers. Then

$$R\dot{I} + \frac{1}{C}I = \dot{V}$$

Suppose that V is constant, $V(t) = V_0$. Solve for $I(t)$, with initial condition $I(0)$.

It is common to write the solution in the form $ce^{-t/\tau}$. Calculate c and τ. Note that τ is measured in the same units as t (in order for the exponent to be dimensionless). It is called the *characteristic time* for the system. What is $I(t + \tau)$ in terms of $I(t)$?

18.03 Differential Equations
Spring 2010