18.03 Class 18, March 14, 2010

Applications in Engineering: A visit by Professor Kim Vandiver

[1] Damping ratio
[2] Measuring the damping ratio
[3] Cell phones on vibrate
[4] Extracting energy from a river

[1] HM: In this unit, we've been studying the equation controlling a spring system

$$m x'' + b x' + k x = F_{ext}$$

We began by thinking about the homogeneous case, the unforced or free system. If the system has any damping, then all these solutions die off; they are transients. This equation is surely too simple to be of any use in an engineering context, isn't it?

KV: I use this every day. This is the oscillator equation. If it makes noise, it vibrates.

How do you describe the solutions?

HM: We factored the characteristic polynomial $p(s) = ms^2 + bs + k$ to analyse this. Factor out the m and complete the square:

$$p(s) = m(s^2 + (b/m) s + (k/m)) = m((s + k/2m)^2 + (k/m - (b/2m)^2)$$

If $k/m > (b/2m)^2$ the roots are imaginary, the system is *underdamped*. Is that an engineering term too?

KV: Yep, and that's the only situation in which you get vibrations. So lets study that case today.

HM: OK. So the root in that case are $-b/2m \pm i \omega_d$

$$\omega_d = \sqrt{k/m - (b/2m)^2} .$$

So the general solution to this homogeneous equation is x_p = A e^{-bt/2m} cos(omega_d t - phi)

Does that look familiar?

KV: Well it's familiar, but now I begin to realize why students look at me and say hunh? I use different notation.

When there's no damping you get the undamped, natural frequency

omega_n = sqrt(k/m)

Let's take omega_d and factor out the quantity omega_n :

omega_d = omega_n (1 - (b/(2 omega_n m))^2)

which I could write as

= omega_n (1 - zeta^2)

where zeta is called the *damping ratio*.

HM: Let's try to re-express the original equation in terms of this zeta . Where's your expression for zeta? Ah, it's there; zeta satisfies

b/(2 m) = omega_n zeta.

So plugging back into the differential equation gives

x" + 2 zeta omega_n x' + omega_n^2 x = 0 .

Now you can see that critical damping occurs when zeta = 1 , underdamped when zeta < 1 . And we can write the solutions in terms of zeta :

x_h = A e^{-omega_n zeta t} cos(omega_d t - phi)

By the way, what are the units of this zeta thing? I mean, omega_n has units 1/sec . What about zeta ?

KV: Well, I'd look at the exponent, which must be dimensionless. Since t is in seconds, omega_n t is already dimensionless, so zeta must be too.

HM: You mean if I switched from metric to feet and hours, the number zeta wouldn't change?

KV: Yep, that's what it means.

[2] Let's try to make a measurement of the damping ratio here. We'll hang a spring, draw the neutral position, draw the spring back, let it go,

and see how long it takes to execute 10 cycles.

The stopwatch reads 14.2 sec . So the period, which I write tau and which is written P in this class, is 1.42 sec/cycle.

So 1/P = damped natural frequency in Hertz = cycles/sec , or about 0.7 Hz.

HM: We've always been talking about circular fequency.

KV: Yes: omega_d = 2 pi (1/P) which comes out to about 4.3 radians/sec.

HM: OK! So we've figured out omega_d by observation. Can we get the system constants omega_n and zeta ?

KV: As a vibration engineer, I have a quick and dirty formula:

    zeta ~ 0.11 / n_50%

where n_50% is the number of cycles it takes for the amplitude to decay by 50%

HM: OK.... well let's measure that.

KV: Let's pull the weight down to here, and count how many cycles it takes till it comes down only half way. By the way, rubber bands are very nonlinear.

HM: Shhh.

KV: Looks like about 4 cycles.

HM: ... so zeta ~ .11/4 ~ .025 ...

KV: Yes. As engineers say, that's two and a half percent damping, 2.5% of critical damping.

HM: So where did this .11 come from? First lets graph this solution. First I'll draw the exponential decay and its negative: +- e^{-zeta omega_n t}.

This gets multiplied by a delayed cosine, $\cos(\omega_d t - \phi)$.
I don't want to try to control what the delay is. The time gap between places where it becomes zero is half the period. But were measuring the bottoms. They don't occur quite where the damped oscillation touches the exponential - the slope isn't zero there. We have to be a little more careful. Think about what happens when I differentiate

$$x_p = A e^{-\zeta \omega_n t} \cos(\omega_d t - \phi)$$

using the product rule. You get two terms. Both will contain $e^{-bt/2m}$.
One will contain $\cos(\omega_d t - \phi)$, the other $\sin(\omega_d t - \phi)$.
But any linear combination of those two sinusoids is another sinusoid of the same circular frequency! so

$$x_p' = A' e^{-\zeta \omega_n t} \cos(\omega_d t - \phi')$$

The distance between places where this is zero is AGAIN half the period, so the time gap between successive minima is the period P .

When t increases by P , what happens to the amplitude $A e^{-\zeta \omega_n t}$?
Ans: it gets multiplied by

$$e^{-\zeta \omega_n P}$$

So after n cycles it gets multipled by $e^{-n \zeta \omega_n P}$. So:

$$1/2 = e^{- \zeta \omega_n P n_{50\%} }$$

Take natural logs:

$$\ln(2) = \zeta \omega_n P n_{50\%} .$$

$$P = 2\pi/\omega_d = 2\pi/(\omega_n \sqrt{1-\zeta^2})$$

When zeta is small, $1 - \zeta^2$ is very close to 1; ω_d is very close to ω_n , so, to good approximation, we can replace ω_d by ω_n.
Put this in:

$$\ln(2) \sim \zeta \omega_n (2\pi/\omega_n) n_{50\%} = 2\pi \zeta n_{50\%}$$

or $\zeta \sim (\ln(2)/2\pi) / n_{50\%}$

And now ln(2)/2pi = 0.1103178 ....

This is the mysterious .11 .

KV: This oscillator equation applies to many things. This string when it vibrates has many modes. The fundamental looks like half a sine wave, which is controlled by the same equation as before, and we can measure its damping ratio.

HM: So the string is controlled by a much more complicated equation, it has infinitely many degrees of freedom and is described by a PDE, but nevertheless its properties can be understood using the simple spring system we've been studying.

KV: So if I pull this back and count the number of cycles to half amplitude .... I count about n_50% ~ 5 ... so what's the damping? zeta ~ .11/5 ~ .02 .

HM: Amazing - so this spring system and this string have something in common - they both have damping of 2% or so.

[3] Now, how about forced vibrations?

KV: How does a cell phone vibrate? Inside there's a little motor, and on the shaft there's an off-center mass. Here's a DC motor with a mass on the shaft. You can see it in the document camera.

A "squiggle pen" works the same way.

Here is a beam, a steel ruler. It has its own natrual frequency, which is determined by its length - it goes like one over length squared - so when you make the length just right the natural frequency would be the same as the frequency of this squigglepen which is attached to the end of it. We can see this with the help of a strobe light. So dim the lights, and I'll keep changing the length of the beam ... till it appears to be permanently bent. You can see the weight, stop action, too.

Now, what if you didn't have a strobe handy, but you wanted to measure the frequency of the squiggle pen.

You can measure it with the ruler, if you can predict the natural frequency of the ruler. There's a formuala for this in terms of the shape of the beam and property of the steel it's made of. It turns out that

ω_n is proportional to $1/L^2$ where L is the length. For this beam,

$\omega_n = 3.706/L^2$.

HM: So it's a ruler.

KV: Yes, it's a frequency ruler. In the old days they had these things marked out in Hertz.

Checking the squiggle pen again, with the strobe, we find resonance - the ruler looks very bent.

HM: Can you bend spoons too?

KV: When you measure the length, you find a frequency.

HM: So if you had the strobe going half as fast, it would look just the same, would't it?

KV: Well you should double the rate of the strobe: you should then see two fixed images.

HM: We've talked a lot about what happens to a fixed system when you vary the frequency of the input signal: you get one of these amplitude response curves. Here what's happening is a little bit different: the input signal (from the squiggle pen) is fixed, and you are varying the system parameters (the length of the ruler).

But still, when the two frequencies get close you see near resonance.

Thank you, Professor Vandiveer. Your research involves vibration, doesn't it?

Yes. I study structures in the ocean. Imagine you have a cylinder (3 feet long, D = 3 inches in diameter), mounted on springs so it can move up and down. Water is flowing past the cylinder with velocity U . In the wake of the cylinder vortices are shed, eddies. The frequency of the vortex shedding is approximately

$$f_S = .2\, U/D$$

If the cylinder were an oil well pipe, D would be about 1 meter;

1 meter per second is a pretty normal speed of water moving in the ocean; so then you expect to see oscillations with frequency .2 Hz.

The vortex shedding causes a force on the cyclinder transverse to the direction of the water flow.

Using this one can extract power from a river. Here is a film that shows a way to do it. Instead of a dashpot, you have a generator sucking energy out of the system.

HM: You wouldn't want to be under one of those things, they look like they have a lot of power.

KV: Yeah, it looks like you could chop carrots with them.

HM: I was thinking of stamping wine ...

Thank you very much, Professor Vandiver.